

4. Reliability and Failure Rates

The term “reliability” in engineering refers to the probability that a product, or system, will perform its designed functions under a given set of operating conditions for a specific period of time. It is also known as the “probability of survival”.

To quantify reliability, a test is usually conducted in which a set of “time-to-failure” sample data is recorded. Let this sample be denoted by $\{t_i, i=1,N\}$; it can then be fitted to a probability density function, $f(t)$ or cumulative function, $F(t)$. Thus the “Reliability” is expressed explicitly by:

$$R(t) = 1 - F(t).$$

The time-dependent behavior of $F(t)$, and hence $R(t)$, of a product may stem from such random factors as inherent defects, loss of precision, accidental over-load, environmental corrosion, etc. Generally, the effect of the various random factors is implicit in the collected sample data $\{t_i, i=1,N\}$; but it is difficult to pin-point **which** one of these factors is predominant and/or **when** it is predominant.

A way to look at the failure behavior in time is to examine the **failure rate**. Failure rate is the **time rate of change of the probability of failure**. Since the latter is generally a function of time, failure rate is also, generally speaking, a function of time. In terms of failure rate, however, one can often obtain some indication as to which of the influencing factors is controlling and at what time it is controlling.

Example 4-1: RCA tested 1000 TV sets in an accelerated reliability evaluation program. In that program, the TV sets were turned on-and-off 16 times every day, mimicking a week of TV usage for a typical family. Based on a failure-to-perform criterion, failure data from the first 10 days of test are:

day-1	day-2	day-3	day-4	day-5	day-6	day-7	day-8	day-9	day-10
18	12	10	7	6	5	4	3	0	1

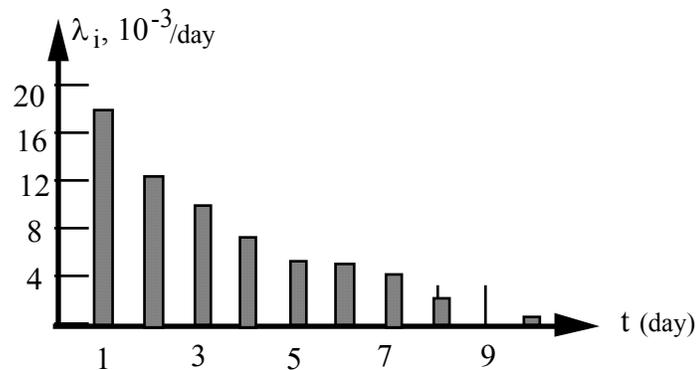
Here, we shall define the failure rate as the "probability of failure per day", denoted by $\lambda_i, i=1, 10$ in the following manner:

For the first day ($i=1$): $\lambda_1=18/1000$ per day;
For the second day ($i=2$), $\lambda_2=12/(1000-18)$ per day.
For the third day ($i=3$) $\lambda_3= 13/(1000-18-12)$ per day.
etc.

Note in the above, that the first day failure rate is based on a total of 1000 TV sets; where as the second day failure is based on a total of (1000-18) sets. Similarly, the failure rate for day-3 is based on a total of (1000-18-12) sets; etc. Clearly, in order for this procedure to yield reliable results, the original number of the TV sets must be large relative to the number of failures.

Note also that the time required in gathering the above data is 10-days, which is a relatively short time period. Yet, as we shall see, the data can already provide useful information about the product.

First, the time behavior of λ_i for the first 10 days can be plotted in a bar chart such shown below:

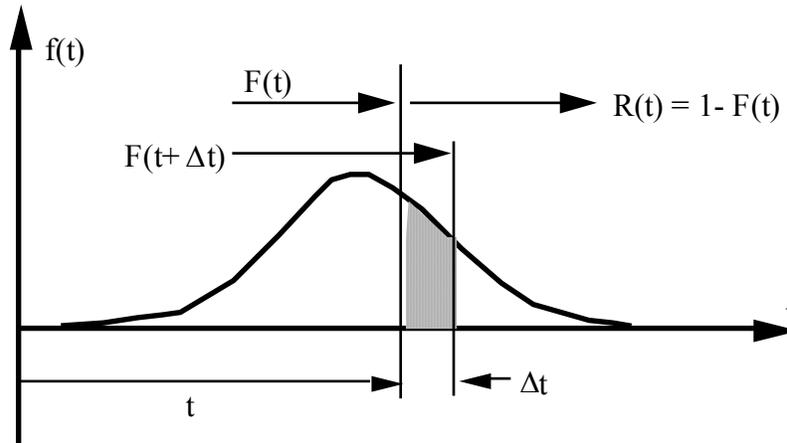


Discussion: The chart shows clearly that the failure rates λ_i are initially high; but they decrease rapidly with time. This exponential-decaying-like behavior is known as **infant mortality** or **wear-in**. It implies that early product failures may be caused by "birth defects", which are inherently present in the product before it is put into service. Products that have survived the wear-in period are deemed not to have the fatal defects at birth, or to have fewer, lesser defects at birth, statistically speaking.

The chart may be fitted by a smooth function $\lambda(t)$ for $0 < t < \tilde{\lambda}$. $\lambda(t)$ is then termed as the failure rate function. As it will soon be shown that $\lambda(t)$ is analytically related to $f(t)$, the time-failure probability function of the same product. In many engineering situations, it is more time-consuming and hence more expensive to determine $f(t)$ than $\lambda(t)$.

4.1 Failure Rate and Failure Probability

Failure Rate. The failure rate function $\lambda(t)$ is analytically related to the time-failure probability distribution, $f(t)$. This relationship can be readily established by examining $f(t)$ graphically, as shown below:



The bell-like curve represents the pdf, $f(t)$; the area under the curve from 0 to t is $F(t)$; the area under the curve from t to ∞ is the **probability of survival**, or the **reliability**, denoted by

$$R(t)=1-F(t).$$

Now, let $\Delta(t)$ be a small time increment from t . Then, consider the fractional probability with failure that occurs within $\Delta(t)$; the latter is represented by the shaded area, $f(t)\Delta t$. Clearly, this fraction of failure could occur only if the product has **survived** the time period from 0 to t . Hence, the probability for the product that fails within Δt is a conditional one, denoted by: $\{f(t)\Delta t\}/R(t)$. The time rate of change of that probability is the failure rate at time t , denoted by $\lambda(t)$; thus

$$\lambda(t) = \{f(t)\Delta t/R(t)\}/\Delta t = f(t)/R(t) \tag{4.1}$$

Eq. (4.1) is the formal relation between $\lambda(t)$ and $f(t)$. In general, one wishes to obtain $f(t)$ when $\lambda(t)$ is known. To this end, we note that

$$f(t) = dF(t)/dt = d[1-R(t)]/dt = -dR(t)/dt$$

and Eq. (4.1) can then be written in the form:

$$\lambda(t) = -[dR(t)]/dt/R(t)$$

or

$$\lambda(t)dt = -dR(t)/R(t)$$

Integration of the above from 0 to t on both sides and noting that $R(0)=1$, we obtain $R(t)$ in explicit term of $\lambda(t)$:

$$R(t) = \exp\left[-\int_0^t \lambda(\tau) d\tau\right] \quad (4.2)$$

Then, from (4.1),

$$f(t) = \lambda(t) \exp\left[-\int_0^t \lambda(\tau) d\tau\right] \quad (4.3)$$

We can readily verify the following, noting that $R(\tau)=0$:

$$\mu = \text{MTTF} = \int_0^{\infty} t f(t) dt = \int_0^{\infty} R(t) dt \quad (4.4)$$

Discussion: The failure rate data (the bar chart) in Example 4-1 can be fitted nicely by the function

$$\lambda(t) = 0.02 t^{-0.56}$$

From (4.3), we obtain the corresponding failure pdf as

$$f(t) = 0.02 t^{-0.56} \exp[-0.04545t^{0.44}]$$

Similarly, from (4.2), we obtain the reliability function:

$$R(t) = \exp[-0.04545t^{0.44}]$$

and the mean-time-to failure:

$$\text{MTTF} = \mu = \int_0^{\infty} \exp[-0.04545t^{0.44}] dt$$

Example 4-2. XYZ company produces and sells video cassette recorders. In order to formulate a pricing, warranty and after-sale service policy, a **reliability testing** program is carried out which finds the CDF for time-to-failure as

$$F(t) = 1 - \exp[-t/8750]$$

where t is expressed in hours.

Now, the associated reliability function is:

$$R(t) = \exp[-t/8750]$$

and the pdf for time-to-failure is:

$$f(t) = \exp[-t/8750]/8750;$$

According to (4.1), the failure rate function is given by:

$$\lambda(t) = f(t)/R(t) = 1/8750 \quad \text{per hour.}$$

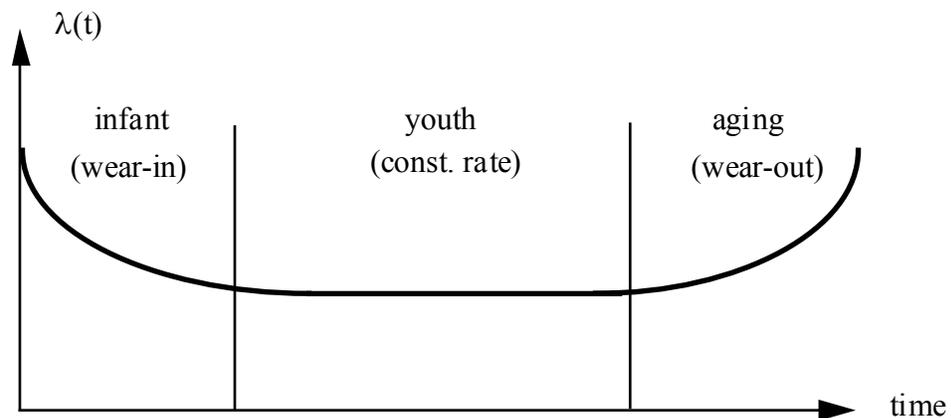
which is a constant.

From using (4.4), the MTTF is given by:

$$\mu=8750 \text{ hours.}$$

Note: In this case, the pdf is an **exponential** function; the corresponding failure rate is a **constant**. The significance of this relationship will be discussed later in this section.

The Bath-Tub Curve. For many engineered products, the failure rate function $\lambda(t)$ has a time-profile much like a **bath-tub** cross-section, such as shown below:

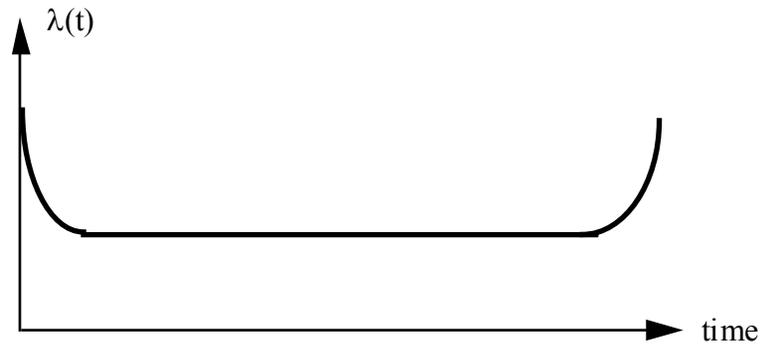


The bath-tub curve is, in fact, an ubiquitous character of all living things as well. For instance, the human life expectancy and the engineered product's failure times may have much in common in their failure rate profiles as portrayed in the bath-tub curve. The bath-tub curve may be broadly classified in three distinct time zones, each corresponds a distinctive **failure mode**.

The **infant mortality** or **wear-in** mode is generally short, with a high but decreasing rate such as in the case discussed in Example 4-1. Engineering wear-in mode may be due to defective parts, defects in materials, damages in handling, out of manufacturing tolerance, etc. These factors show their effects early in life, resulting in wear-in mode. To correct this situation, one may resort to design improvement, care in materials selection and tightened production quality control. When such measures are insufficient, a **proof-test** of the products may be instituted; i.e. all products under go a specified period of simulated test, in which most early failures are detected. There is another measure:

redundancy may be built into the product so as to provide a **fail-safe** feature.

The **youth** or **constant rate** mode is exhibited by those product that have survived the wear-in period. The rate is generally the lowest; and in some product it maintains a long and flat behavior such as shown below:

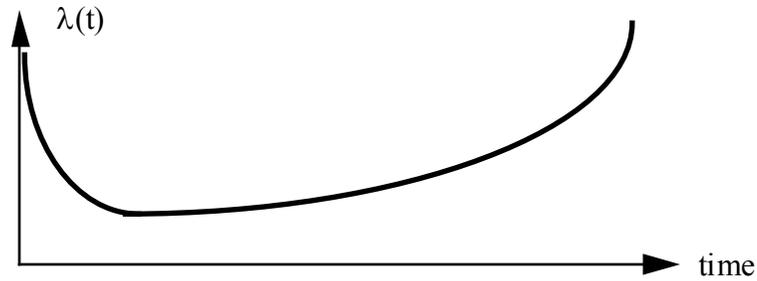


This failure mode is generally caused by **random events** from without, rather than by inherent factors from within. In the youth period, the rate is constant and the associated pdf is one of exponential function (see, Example 4-2).

Random failure can be reduced by improving product design, making it more robust with respect to the service condition to which it is exposed in real life. From the product sales point of view, the constant rate period is often used to formulate the pricing, warranty and servicing policies. This aspect is particularly important in consumer goods such as lap-top computers, cell phones, etc.

As it will be shown later, products with a constant failure rate has the unique attribute that the probability of failure in future time is independent of the product's past service life, regardless how long the service life is. This aspect will be further explored later on in discussing product repair frequency, spare-part inventory and product maintenance in general.

The **aging** or **wear-out** mode is usually due to material fatigue, corrosion, embrittlement, contact wear at joints, etc. The wear-out mode is often encountered in mechanical systems with moving parts such as valves, pumps, engines, cutting tools, bearing balls in joints, automobile tires, just to mentioned a few. Onset of rapidly increasing rate requires measures such as increased regularity of inspection, maintenance, replacement, etc. since in these products the youth period is relatively short while the wear-out period is long, such as depicted below:



The reason for this kind of behavior is that aging effects can set in early in service life; they precipitate themselves in time and cause eventual failure of the product. Thus, the central concern in wear-out is to predict or estimate the product's probable service life. With that information, either a sufficiently long life is designed into the product at the outset; or a prudent scheduling in preventive maintenance is practiced after the product is put into service.

Generally speaking, the wear-in mode is a **quality-control** issue, while the wear-out mode is a **maintenance** issue. The random failure or constant rate mode, on the other hand, is widely used as the basis for product **reliability** considerations. Some of the key features in and applications of the latter are discussed in the next section.

4.2 Constant Failure Rate Reliability Models

Single Product with Constant Rate. If the failure rate is constant for all times, $\lambda(t) = \lambda_0$, the corresponding reliability is an exponential function given by (4.2):

$$R(t) = \exp(-\lambda_0 t) \quad (4.5)$$

The pdf $f(t)$ is given by (4.3):

$$f(t) = \lambda_0 \exp(-\lambda_0 t) \quad (4.6)$$

The mean (mean-time-to-failure) and the standard deviation are readily obtained as

$$\begin{aligned} \mu &= \text{MTTF} = 1/\lambda_0 \\ \sigma &= \mu = 1/\lambda_0 \end{aligned} \quad (4.7)$$

It is noted that at the mean-time-to-failure, $t=\mu=1/\lambda_0$, the reliability is given by

$$R(\mu) = \exp(-1) = 0.368; \quad \text{or} \quad F(\mu) = 0.632.$$

Example 4-3. A device in continuous use has a constant failure rate $\lambda_0=0.02/\text{hr}$. The following may be computed:

(a) the probability of failure within the first hour of usage:

$$P\{t \leq 1\} = F(1) = 1 - \exp(-0.02 \times 1) = 1.98\%.$$

(b) the probability of failure within the first 10 hours:

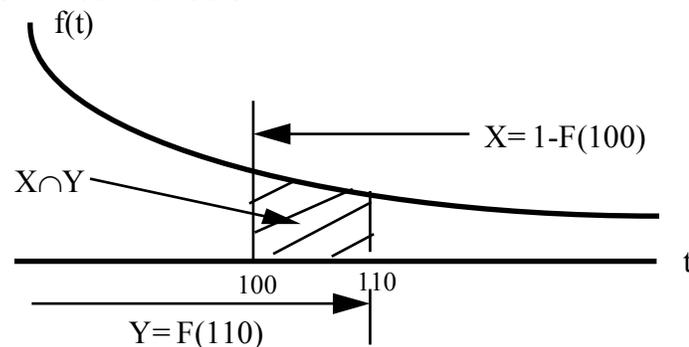
$$P\{t \leq 10\} = F(10) = 1 - \exp(-0.02 \times 10) = 18.1\%.$$

(c) the probability of failure within the first 100 hours:

$$P\{t \leq 100\} = F(100) = 1 - \exp(-0.02 \times 100) = 86.5\%.$$

(d) the probability of failure within the next 10 hours, if it has already been in use for 100 hours.

This is conditional to the fact that the device has already survived 100 hours. As illustrated in the graph shown below, we designate X = that survived 100 hours; Y = that survived 110 hours:



Then, the answer to (d) is thus:

$$P\{Y/X\} = P\{X \leftrightarrow Y\} / P\{X\} = [F(110) - F(100)] / [1 - F(100)]$$

Since from above calculations, we already found,

$$F(110) = 1 - \exp(-0.02 \times 110) = 88.92\% \text{ and} \\ F(100) = 86.5\%$$

Hence,

$$P\{Y/X\} = (0.8892 - 0.865) / (1 - 0.865) = 18.1\%.$$

Discussion: The result in (d) is identical to the result in (b). The device, being of a constant failure rate, has no memory of prior usage. Specifically, within any fixed period of usage, the probability of failure is the same.

Single Product Under Repeated Demands. Suppose that a product is called to service by “demand” (e.g. turning on of a water pump) and that the probability of failure for responding to the demand is p . If N is the number of demands called during the time period t , we define the average number of demands per unit of time as

$$m = N/t \quad (4.8)$$

Assuming that failure by demand is an independent event, the reliability of the unit subjected to N repeated demands is (see Chapter III, the binomial distribution)

$$R_N = (1-p)^N.$$

If $p \ll 1$ and $N \gg 1$, we reduce R_N in the form of Poisson distribution:

$$R_N \approx e^{-Np} = e^{-mp} t \quad (4.9)$$

Now, if we set

$$\lambda_0 = mp \quad (4.10)$$

Eq. (4.9) to the form of (4.5), the constant rate reliability function.

Hence, the reliability of a product under repeated demands is a case of “constant failure rate”, provided that $p \ll 1$ and $N \gg 1$.

Example 4-4. Within one-year warranty period, a cell phone producer finds that 6% of the phones sold were returned due to damage incurred in accidental dropping of the phone on hard floor. A simulated laboratory test determined that when a phone is dropped on hard floor the probability of failure is $p=0.2$.

Based on this information, the engineers at the phone manufacturer made the following interpretation:

(a) Let the time unit be “year”. Then, for a single unit:

$F(1)$ = cumulative probability of being damaged within 1 year = 0.06; Or, $R(1) = 0.94$ = the reliability up to 1 year.

The above can also be interpreted as 6 phones out of every 100 were damaged per year; Or, 94 phones out of every 100 did not suffer any damage during the year.

(b) Let m = number of demands (drops on hard floor) per phone per

year. Then, from (4.9):

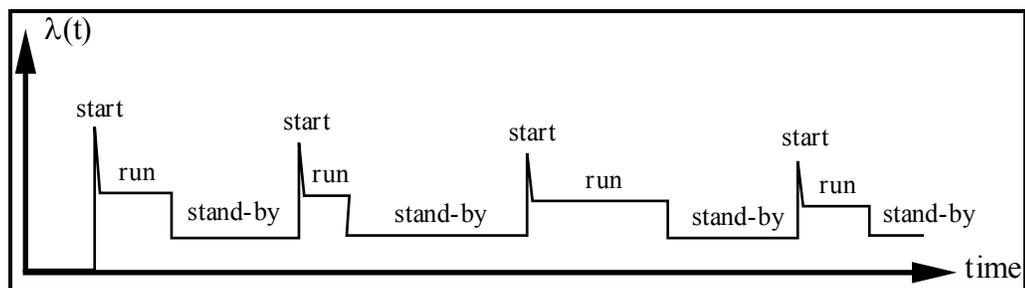
$$R(1) = 1 - F(1) = \exp(-mpt) = \exp(-m \times 0.2 \times 1) = 0.94$$

Solving the above, we obtain $m=0.31$ drop per phone per year.

Interpretation: on the average, there are 31 drops per 100 phones per year. Or, alternatively, the mean time between failure (drop) of one phone is: MTBF = $1/0.31 = 3.23$ yrs.

Discussion: Given that $m=0.31$ is a factor stemming from customers habits, the phone producer can only redesign the phone by making it more impact resistant; this will decrease the value of p . For instance, if $p=0.1$, then $R(1) = \exp(-0.31 \times 0.1 \times 1) = 0.97$; or $F(1) = 0.03$. It cuts the returning rate from 6% to 3% per year.

Step-Wise Constant Rate. Many operating systems may be treated as having “step-wise constant failure rate”. As an illustration, consider the electric motor used in a household heat-pump system. When the room-temperature is low, the motor is called to drive the heat-pump until the room temperature is raised to the preset high; the motor is then turned off in the stand-by state. During a typical service cycle, say 24 hour, the motor may be called into service N times; and this is depicted schematically by the graph shown below:



To evaluate the reliability of this motor, the following input information is required:

- N = the number of starts (demands) per service cycle (24 hours);
- c = time fraction when the motor is in running during the service cycle;
- $1-c$ = time fraction when the motor is in stand-by state, during the service cycle;
- p = probability of failure when the motor responds to a call (start)
- λ_r = failure rate (per hour) when motor is in the running state; and

λ_s = failure rate (per hour) when motor is in the stand-by state.

A “combined” or “equivalent” failure rate, λ_c , for the motor in service can be expressed as:

$$\lambda_c = \lambda_d + c \lambda_r + (1-c)\lambda_s \quad (4.11)$$

where $\lambda_d = mp$, m being the number of calls per unit of time (i.e. $N/24$ calls per hour).

Therefore, the reliability function of the motor is given by:

$$R(t) = \exp(-\lambda_c t) \quad (4.12)$$

Clearly, in order for (4.12) to be accurate, the service time t should be much greater than just one single cycle (24 hours).

Example 4.5. An electric blower is used in a heating system. The manufacturer of the blower has provided the following failure rate data:

- p = failure probability on demand = 0.0005 per call;
- λ_r = failure rate per hour when blower is in running = 0.0004/hr;
- λ_s = failure rate per hour when blower is in stand-by = 0.00001/hr.

During the a typical 24 hours in the winter months, the following data is obtained from the heater’s operation recording:

# of calls	time of call	time of stop running	running time
1	0:47 am	1:01 am	0.23 hr
2	1:41 am	2:07 am	0.43 hr
3	2:53 am	3:04 am	0.18 hr
4	3:55 am	4:13 am	0.30 hr
5	4:43 am	5:05 am	0.37 hr
6	5:58 am	6:19 am	0.35 hr
7	6:50 am	7:14 am	0.40 hr
8	7:46 am	8:07 am	0.35 hr
9	8:55 am	9:08 am	0.22 hr
10	9:49 am	10:05 am	0.27 hr
11	10:49 am	11:01 am	0.20 hr
12	11:52 am	12:08 pm	0.27 hr
13	12:59 pm	1:11 pm	0.20 hr
14	1:49 pm	2:04 pm	0.25 hr

Now, consider the time interval from t to $t+\Delta t$. First, let us consider the probability that zero repair will occur before t and denote it by $p(0/t)$. Similarly, the probability that zero repair will occur before $t+\Delta t$ is noted by $p(0/t+\Delta t)$. Note then, in order for $p(0/t+\Delta t)$ to occur, we must first have $p(0/t)$. We wish to obtain the exact expression for both.

Now, if the device has a constant failure rate λ_o , the failure probability **during** Δt is $\lambda_o\Delta t$ while the non-failure probability is $(1 - \lambda_o\Delta t)$. Consequently, we can write:

$$p(0/t+\Delta t) = p(0/t)\supseteq(1 - \lambda_o\Delta t)$$

Rearranging and letting $\Delta t \searrow 0$, we obtain the differential relation:

$$\delta p(0/t)/\delta t = -\lambda_o p(0/t)$$

Integrating the above over the range from $t=0$ to t , and noting the conditions in (4.12), we find the probability for zero repair within the time period of t :

$$p(0/t) = \exp(-\lambda_o t) \tag{4.14}$$

which is in the form of an exponential function.

To find the expression for $p(n/t)$, we consider the probability that n (>0) repairs will occur over the time t ; thus, we determine first the probability that n repairs occur over the time period $t+\Delta t$. This will happen in two different situations:

- (1) n repairs occur already over t ; hence, no more repair occurs during Δt ;
- (2) $\underline{n-1}$ repairs occur over t ; then one repair must occur during Δt (noting

$\Delta t \searrow 0$).

Hence, we can write:

$$p(n/t+\Delta t) = p(n/t)\supseteq(1 - \lambda_o\Delta t) + p(\underline{n-1}/t)\supseteq\lambda_o\Delta t$$

The above can be rewritten in the differential form:

$$\delta p(n/t)/\delta t = -\lambda_o p(n/t) + \lambda_o p(\underline{n-1}/t) \tag{4.15}$$

Integration of (4.15) from 0 to t yields the following integral expression for $p(n/t)$:

$$p(n/t) = \lambda_o \exp(-\lambda_o t) \int_0^t p(\underline{n-1}/\tau) \exp(-\lambda_o \tau) d\tau \tag{4.16}$$

Equation (4.16) is a recursive relationship; it allows for the determination of $p(n/t)$ successively for $n=1,2,3, \dots$. For instance, for $n=1$, we can substitute the result in (4.14) into (4.16) and carry out the integration, obtaining:

$$p(1/t) = (\lambda_0 t) \exp(-\lambda_0 t) \quad (4.17)$$

and for $n=2$, we in turn obtain:

$$p(2/t) = [(\lambda_0 t)^2/2] \exp(-\lambda_0 t) \quad (4.18)$$

In fact, a general and explicit expression for $p(n/t)$ in (4.16) is given by:

$$p(n/t) = [(\lambda_0 t)^n/n!] \exp(-\lambda_0 t) \quad (4.19)$$

We note that (4.19) has exactly the form of the **Poisson distribution** for n (see, Eq. 2-21), whose mean and variance has the same value (see Example 2.11):

$$\mu_n = \sigma_n^2 = \lambda_0 t \quad (4.20)$$

where μ_n is known as the “mean number” of repairs needed over the time period t .

Mean-Time-Between-Failure. With the "mean number of failure" over t given by (4.20). the "mean time between failure", or MTBF for short, of the "maintained" device is defined as:

$$\text{MTBF} = t/\mu_n \quad (4.21)$$

As it turns out, for constant failure rate, the MTBF for the “maintained” device is

$$\text{MTBF} = 1/\lambda_0$$

which is the same as the MTTF of the single device whose $f(t)$ is an exponential function given by (4.6).

With the Poisson distribution in (4.19), the **cumulative probability** that **more than N** repairs are needed over the designated time period t is given as follows:

$$P \{(n>N)/t\} = \sum_{n=N+1}^{\infty} [(\lambda_0 t)^n/n!] \exp(-\lambda_0 t) = 1 - \sum_{n=0}^N [(\lambda_0 t)^n/n!] \exp(-\lambda_0 t) \quad (4.22)$$

Note that the above is an "cumulative" probability. A more precise interpretation of the terms in the equation is as follows: (a) the term which sums from $n=0$ to $n=N$ represents the probability that **up to N** repairs will occur during the time period t ; and (2) the term which sums from $n=N+1$ to $n \rightarrow \infty$ represents the probability for **more than N** repairs will occur during the time period t . Clearly, the sum of the two terms represents the total probability of a unity (see Eq. 4.13).

Example 4.6. A DC power pack (in a computer) is in continuous use; and it has a constant failure rate $\lambda_0=0.4$ per year. If a spare is kept on hand in case of the power pack failure, what is the chance of running out of the replacement spare within a 3-months period?

Solution: Here, the designated time period is $t=1/4$ year; the mean number of failures in 3 months is $\lambda_0 t=0.1$. The probability that **more than 0** (or at least 1) failure occurs within $t=1/4$ is calculated from using (4.22) for $N=0$:

$$P\{\underline{n>0}/t\} = 1 - \exp(-\lambda_0 t) = 0.095$$

Or, there is roughly 10% chance that the spare will be used within 3 months.

Discussion: If 2 spares are kept in hand, the chance of running out of the spares within 3-months would be calculated also from (4.22). Thus, by setting $N=1$ (i.e. more than 1, or at least 2), we obtain:

$$P\{\underline{n>1}/t\} = 1 - (1+\lambda_0 t) \exp(-\lambda_0 t) = 0.00468.$$

which is less than one half of 1%.

Example 4.7. It is known that the MTBF of a truck tire due to puncture is about 150 Km on the run. A truck with 10 wheels always carries some spare tires.

- (a) What is the chance that at least 1 spare will be used on a 10 Km trip?
- (b) What is the chance at least 2 spares are needed on a 10 Km trip?
- (c) How many spare tires should the truck keep in order to have more than 99% assurance that it will not run out of spares on a 10 Km trip?

Solution: Puncture of tire is a random event; hence, it can be treated as having a constant failure rate. In this example, the $MTBF=150$ Km is given in apriori. This information is usually determined by running a large data base. For instance, if the number of tire punctures n is large and it is gathered over a long period of time t , we can approximate the $MTBF \blacktriangleright t/n$.

Now, since the truck has 10 wheels, the 10 tires will accumulate a total of $t=100$ Km over a trip of 10 Km. Given the $MTBF= 150$ Km and $\lambda_0=1/MTBF$, we calculate $\lambda_0 t = 100/150 = 0.667$. Hence, the probability that:

$$(a) \text{ at least 1 punctured tire: } P\{\underline{n>0}/t\} = 1 - \exp(-\lambda_0 t) = 0.487;$$

$$(b) \text{ at least 2 punctured tires: } P\{\underline{n>1}/t\} = 1 - (1+\lambda_0 t) \exp(-\lambda_0 t) =$$

0.144;

(c) for 99% assurance that it will not run out of spares, we determine the N value from:

$$P\{(n>N)/t\} \leq 1\%$$

Thus, for N=2 (for at least 3 punctured tires), we have

$$P\{(n>2)/t\} = 1 - [1 + \lambda_0 t + (\lambda_0 t)^2 / 2] \exp(-\lambda_0 t) = 0.03$$

and the probability of running out of spares is larger than 1%.

For N=3, for at least 4 punctured tires, we find

$$P\{(n>3)/t\} = 1 - [1 + \lambda_0 t + (\lambda_0 t)^2 / 2 + (\lambda_0 t)^3 / 6] \exp(-\lambda_0 t) = 0.00486.$$

Hence, for 99% assurance or better, 4 spares should be kept.

4.3 Time-Dependent Failure Rates

(optional)

Based on the general characteristics of the bath-tub shape for failure rates, the infant-mortality and the aging effects are associated with time-dependent failure rates. The former is associated with a rate that decreases with time; and the latter is associated with a rate that increases with time. These are commonly referred to as the wear-in and wear-out phenomena, respectively.

Wear-In Mode of Failure. The failure rate data plotted in Example 4-1 exhibits the “wear-in” behavior; one can fit the plot with a decreasing function $\lambda(t)$:

$$\lambda(t) = a t^{-b} \tag{4.23}$$

where both a and b are positive, real-valued constants determined in fitting the data.

Using $\lambda(t)$, the reliability function R(t) and the pdf f(t) for time-to-failure are found by integrating (4.2) and (4.3), respectively. After that, a number of reliability questions may be answered.

Example 4-8. A circuit has the decreasing failure rate described by $\lambda(t)=0.05/t$ per year. Here, $\lambda(t)$ has the form of (4.23). The associated reliability function R(t) and the pdf f(t) are integrated from (4.2) and (4.3),

respectively:

$$R(t) = \exp[-0.1 |t|]; \quad f(t) = 0.05 \exp[-0.1 |t|] / t$$

with the above, the following are computed:

- (a) the reliability for 1 year use is $R(1) = \exp[-0.1] = 0.905$; or the failure probability is $F(1) = 9.5\%$;
- (b) the reliability for the first 6-month use is $R(0.5) = \exp[-0.1 |0.5|] = 0.93$.
- (c) after 3 years, the fraction of failed circuits is $F(3) = 1 - R(3) = 1 - \exp[-0.1 |3|] = 0.16$; or 84% still in use.
- (d) suppose that a circuit has been in service for 1 year without failure; the reliability for it to be in service for an additional 6-month is $R(0.5/1)$. To find $R(0.5/1)$, we first calculate $F(0.5/1)$:

$$F(0.5/1) = [F(1.5) - F(1)] / R(1) = [0.115 - 0.095] / 0.905 = 0.022$$

We then have

$$R(0.5/1) = 0.98$$

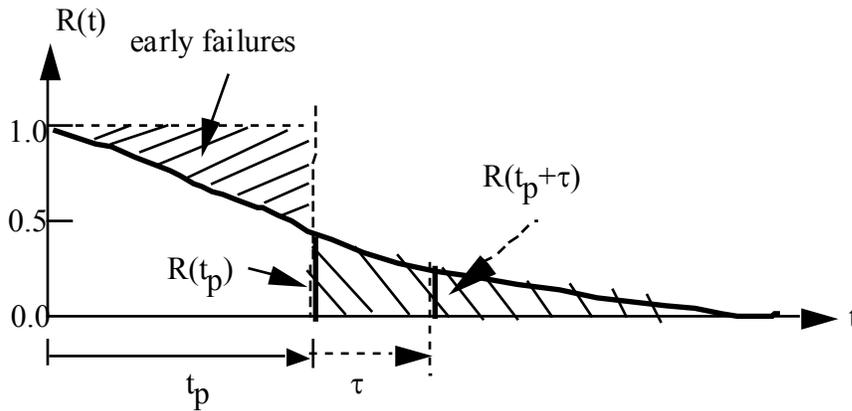
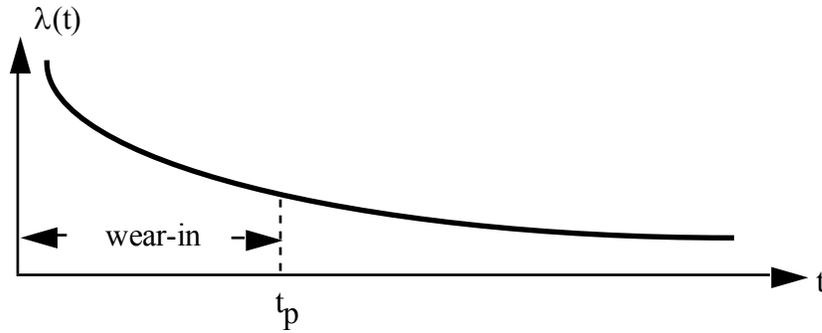
which is higher than $R(0.5)$ computed in (b) above.

Discussion: The result in (d) shows that if early failures are eliminated, the remaining circuits will have a higher reliability. This gives rise to the concept of **proof-test**, a practice often used in quality-control of engineering products.

The Concept of Proof-Test. Engineering products with a "short" wear-in time but high rate may be subjected to a **proof-test**. The idea is to screen out early failures, so that the products which survived the proof-test will be in their "constant rate" mode, thus yielding a much higher reliability.

In practice, a proof-test often involves a simulation of the product in service for a period of wear-in time, t_p ; but the choice of t_p depends on the nature of failure of the product, or more precisely, the behavior of the failure rate function, $\lambda(t)$.

The following sketches show the basic elements involved in proof testing, including the failure rate function $\lambda(t)$ and the corresponding reliability function $R(t)$:



During the proof-test period from 0 to t_p , some early failures may have occurred (the upper shaded area) and some may have survived (the lower shaded area).

Now, let τ be the service time for those past the proof-test; then the reliability of the product after proof-test is given by:

$$R(\tau/t_p) = R(t_p + \tau)/R(t_p) = \exp[-\int_0^{t_p + \tau} \lambda(\xi)d\xi] / \exp[-\int_0^{t_p} \lambda(\xi)d\xi]$$

Upon rearranging,

$$R(\tau/t_p) = \exp[-\int_{t_p}^{t_p + \tau} \lambda(\xi)d\xi]; \quad \tau > 0 \quad (4.24)$$

Clearly, the reliability of the product after the proof-test, $R(\tau/t_p)$, is much improved.

Discussion: The case in Example 4-8(d) is one related to proof-testing. We can now use (4.24) to compute the wanted result. We find $R(0.5/1) = 0.98$ which is the same as in Example 4-8(d).

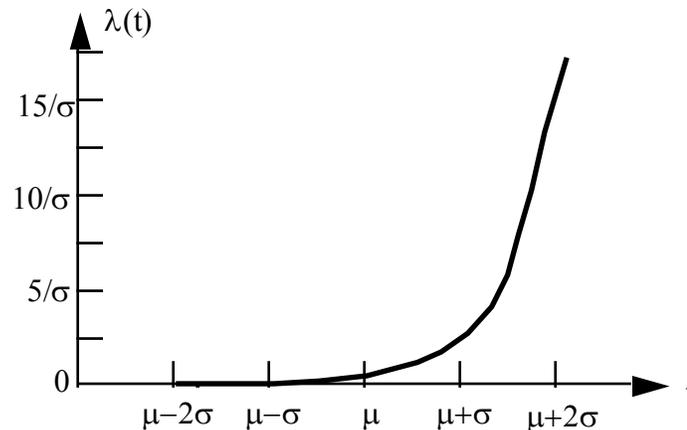
Normal Distribution and Failure Mode. The failure mode described by the popular normal distribution is one of the “wear-out” mode. This is readily obtained from the pdf for the normal distribution is given in (3.11); the corresponding reliability function can be expressed as $R(t)=1-\Phi(z)$, where $\Phi(z)$ is the standardized CDF via the transformation:

$$z=(t-\mu)/\sigma$$

It follows from (4.1) that the failure rate function is given by:

$$\lambda(t) = (1/\sqrt{2\pi})(1/\sigma) \exp[-z^2/2]/[1-\Phi(z)] \quad (4.25)$$

With the help of Appendix III-A, (4.25) can be plotted for $\lambda(t)$ versus the time t :



Note that $\lambda(t)$ rises exponentially when time t exceeds the MTTF (i.e. beyond $t = \mu$).

Example 4-9. Field data from a tire company shows that 90% of the tires on passenger cars fail to pass inspection between 22 to 30 k-miles. The data also shows that the time-to-failure probability of the tires can be described by a **normal** distribution.

- What is the failure rate when a tire has 20 k-miles on it?
- What is the failure rate when a tire has 25 k-miles on it?

Since 90% of the tires fail between 20 and 30 k-miles or the central population is 90%, we can write:

$$\Phi(z_{20k}) = 0.05 \qquad \Phi(z_{30k}) = 0.95$$

Using Appendix III-A, we find,

$$z_{20k} = (20-\mu)/\sigma = -1.65 \quad z_{30k} = (30-\mu)/\sigma = 1.65$$

Solving the above for

$$\mu = 25 \text{ k-miles} \quad \text{and} \quad \sigma = 3.03 \text{ k-miles}$$

Having found the values of σ and μ , the pdf $f(t)$ is totally defined. The corresponding failure rate function $\lambda(t)$ is then computed from (4.25). Thus,

- (a) for a tire having 20 k-miles on it, $\lambda(t=20)=0.14$ per k-miles; and
- (b) for a tire having 25 k-miles on it, $\lambda(t=25)=0.8$ per k-miles.

Since λ increases with t (miles), failure of the tire is in a wear-out mode.

Log-Normal Distribution and Failure Mode. The pdf for the log-normal distribution, $g(t)$ is explicitly expressed in (3.33). The geometric behavior of $g(t)$ is one of “left-skewed” function, dictated by the parameters ω_0 and t_0 (especially ω_0). In particular, when the value of ω_0 approaches 1, $g(t)$ is nearly exponentially distributed; when ω_0 approaches 0.1, $g(t)$ is nearly normally distributed. Thus, the failure rate function corresponding to $\omega_0 \nearrow 1$ is one of “**constant rate**” mode while the failure rate function corresponding to $\omega_0 \nearrow 0.1$ is one of “**wear-out**” mode. The latter is somewhat a slower increasing function of t , however.

According to (4.1) and with the substitution of (3.33), we obtain the failure rate function for the log-normal distribution:

$$\lambda(t) = \left\{ \frac{1}{\omega_0 \sqrt{2\pi}} \frac{1}{t} \exp \left[-\frac{1}{2} \left(\frac{\ln(t/t_0)}{\omega_0} \right)^2 \right] \right\} / [1 - \Phi(z)] \quad (4.26)$$

where $\Phi(z)$ is the standardized CDF of $g(t)$ via the transformation (see Eq. 3.35):

$$z = [\ln(t/t_0)]/\omega_0 \quad (4.27)$$

Example 4-10. Failure of a shock-absorber used in passenger cars is described by the log-normal function. Field data shows that 90% of the shock absorbers fail between 120 k-miles and 180 k-miles.

What is the failure rate of the shock absorber at $t = 150$ k-miles?

Solution: Given that $g(t)$ is log-normal, we need to determine the parameters t_0 and ω_0 from the given field data. To that end, we start from the standardized CDF of $g(t)$: $\Phi(z)$. From the field data, we have

$$\Phi(z_{120}) = 0.05 \quad \text{and} \quad \Phi(z_{180}) = 0.95$$

Using Appendix III-A and (4.27), we find:

$$z_{120} = \ln(120/t_0)/\omega_0 = -0.1645$$

$$z_{180} = \ln(180/t_0)/\omega_0 = 0.1645$$

From there, we solve for t_0 and ω_0 :

$$t_0 = 147 \text{ k-miles} \quad \text{and} \quad \omega_0 = 0.1232$$

With t_0 and ω_0 , we have determined $g(t)$. Then, the failure rate function is given by (4.26).

Thus, at $t = 150$ k-miles, we compute $\lambda(t=150) = 0.49$ per k-miles. It is readily obtained that $\lambda(t_0) = \lambda(t=147) = 0.044$ per k-miles; $\lambda(t=120) = 0.009$ per k-miles, etc. We see that the failure rates increase rapidly with time (k-miles). This is so because $g(t)$ is almost normal-like, as the value of ω_0 is only 0.1232 which is close to being 0.1.

Discussion: In general, the failure rate will increase sharply once t is greater than t_0 . Note that when $\omega_0 \ll 1$, the log-normal function reduces to being exponential; and the failure rate will be constant in time (see Chapter III, section III-4).

Weibull Distribution and Failure Modes. The failure rate function for the Weibull distribution is easily obtained from using (3.37) and (4.1):

$$\lambda(t) = (m/\theta)(t/\theta)^{m-1} \tag{4.28}$$

The failure mode represented by (4.28) depends on the value of m . When $0 < m < 1$, $\lambda(t)$ is a decreasing function of t , representing the wear-in mode; when $m=1$, the Weibull reduces to exponential and $\lambda(t)$ is constant in t , representing the random failure mode; when $m \geq 2$, $\lambda(t)$ is an increasing function of time, representing the wear-out mode (see Example 4.11 below).

Example 4-11. A hearing aid has the time-to-failure pdf in the form of a Weibull distribution:

$$f(t) = (m/\theta)(t/\theta)^{m-1} \geq \exp[-(t/\theta)^m]$$

with the shape parameter $m=0.5$ and the scale parameter $\theta=180$ days.

Since the Weibull CDF, $F(t)=1-\exp[-(t/\theta)^m]$, hence the reliability function is: $R(t)=\exp[-(t/\theta)^m]$. The corresponding failure rate function is given by (4.1). Thus, we have

$$\lambda(t) = f(t)/R(t) = (m/\theta)(t/\theta)^{m-1}$$

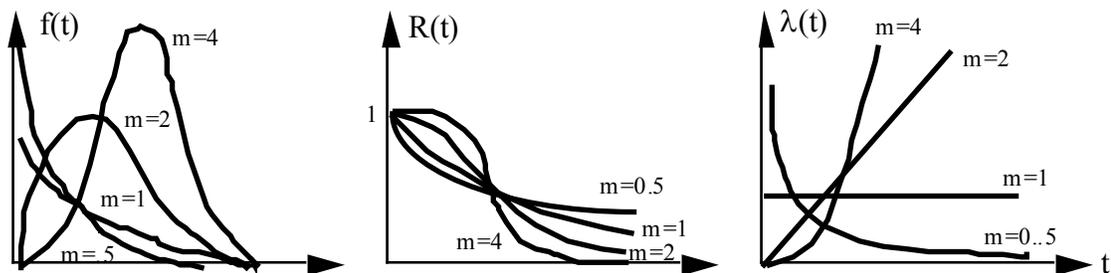
For $m=0.5$ and $\theta=180$ days,

$$\lambda(t) = 0.0373 t^{-0.5} \text{ per day.}$$

which is in the form of (4.23), representing the wear-in failure mode.

If we set $m=1.5$, we obtain $\lambda(t) = 0.00062 t^{0.5}$ which is an increasing function of time t .

Discussion: The parameter m in the Weibull function controls the shape behavior of $f(t)$ and hence the reliability function $R(t)$ and the failure rate function $\lambda(t)$. The figure below shows the interrelations linking $f(t)$, $R(t)$ and $\lambda(t)$ for $m = 0.5, 1.0, 2.0$ and 4.0 :



Note: When $m < 1$, the Weibull function corresponds to a wear-in mode of failure; when $m = 1$, $f(t)$ is an exponential function with $\lambda(t)$ a constant ($=1/\theta$); this, of course, represents a random failure mode as in the “youth” period. When $m > 1$, $f(t)$ becomes increasingly a left-skewed while $\lambda(t)$ becomes an ascending function of t ; this represents a wear-out failure mode. When $m \gg 4$, $f(t)$ resembles that of the normal function, which is in a severely wear-out mode.

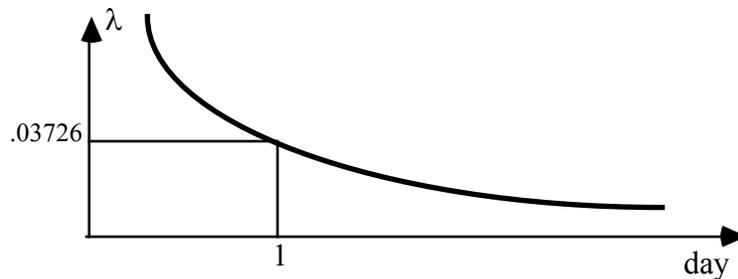
Example 4-12. A hearing-aid manufacturer finds that the product has a high scatter of quality; its pdf for time-to-failure can be described by the Weibull

distribution with $m=0.5$ and $\theta=180$ days. Let us examine the properties of this hearing-aid in the following situations:

First, from (4.28), the failure rate function for the hearing-aid is given explicitly by

$$\lambda(t)=0.03726(t)^{-0.5}$$

The failure mode of the hearing-aid is one of wear-in type. Note that λ at day-1 is 0.03726 per day and it decreases rapidly as days go on. A graph of $\lambda(t)$ is shown below:



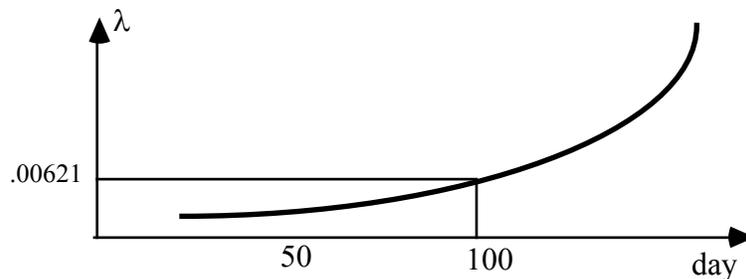
Second, let us say that the manufacturer has conducted a proof-test on the hearing aid so as to screen out any infancy failures; those that passed the test will exhibit a better quality and with a smaller variability in quality.

Let us say that the pdf for time-to-failure of the screened hearing-aid can now be described by another Weibull distribution with $m=1.5$ and $\theta=180$ days. Then, according to (4.28), the failure rate function is now:

$$\lambda(t)=6.21 \times 10^{-4}(t)^{0.5}$$

The mode of the failure rate has now changed to an increasing function of t , representing the wear-out mode. Note that the failure rate at day-1 is now only 0.000621 per day but it is rapidly increasing as the days go on.

The corresponding graphic for $\lambda(t)$ is shown below:



From the above examples, it is seen that the Weibull distribution is so versatile as to describe all the three different failure modes (the wear-in, constant-rate and wear-out modes) which comprise the entire bath-tub curve. For this, a combined failure rate function in the general form is proposed as follows:

$$\lambda(t) = (m_a/\theta_a)(t/\theta_a)^{m_a-1} + (m_b/\theta_b)(t/\theta_b)^{m_b-1} + (m_c/\theta_c)(t/\theta_c)^{m_c-1} \quad (4.29)$$

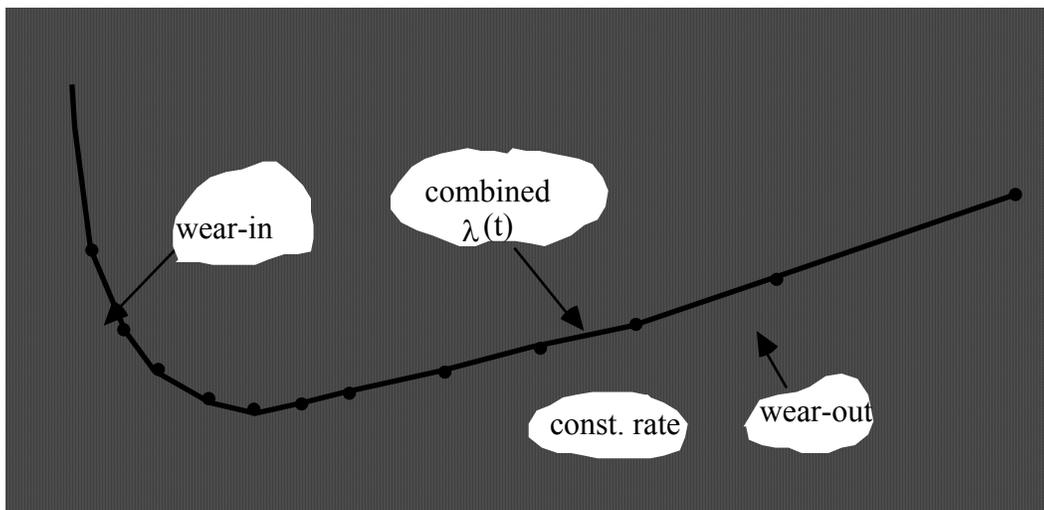
where

$$0 < m_a < 1, m_b = 1 \text{ and } m_c > 1; \text{ also } \theta_a < \theta_b < \theta_c.$$

Note that the first term in (4.29) is decreasing with t , which is in the infantile mode; the second term is constant in t , hence in the youth mode; and the last term is increasing with t , a wearing out mode. Proper choice of the parameters m 's and θ 's in (4.29) will yield a smooth bath-tub curve for $\lambda(t)$, as illustrated by the example below.

Example 4-13. The failure rate function for a cutting knife is known to be bath-tub-like. The wear-in portion may be described by $\lambda(t) = (m_a/\theta_a)(t/\theta_a)^{m_a-1}$ with $m_a = 0.2$ and $\theta_a = 10$ (days); the constant rate portion by $\lambda(t) = 1/100$ per day; and the wear-out portion by $\lambda(t) = (m_c/\theta_c)(t/\theta_c)^{m_c-1}$ with $m_c = 2.5$ and $\theta_c = 120$ days. Plot the bath-tub curve and comment on it.

Solution: $\lambda(t) = 0.1262(t)^{-0.8} + 0.01 + 1.58 \times 10^{-5}(t)^{1.5}$ per day, which is graphically shown below:

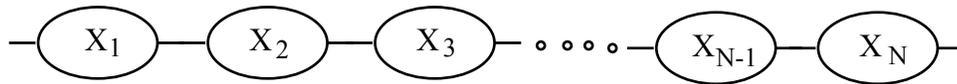


Discussion: The failure mode of this cutting knife does not seem to possess a constant-rate character. The wear-in period is short (about 25 days) while the wear-out period is long.

4.4 System Failure Rates

An engineering system is, generally, a combination of numerous sub-components; each of these components can fail during the system's service life. Failure of a certain component may or may not cause failure of the overall system; but, even if it does not, it can reduce the degree of reliability of the system. For purpose of analysis, the system is often represented as a combination of two basic models: the "in-series" and the "in-parallel" models.

The In-Series Models. Suppose that the system S contains N sub-components, denoted by $X_i, i=1, N$; and they are "linked" in a series as depicted graphically below:



Let $f_i(t)$ be the pdf for the time-to-failure of X_i ; then the corresponding failure rate is $\lambda_i=f_i(t)/R_i(t)$; see (4.1). Or, from (4.2), we can express the reliability function for X_i as:

$$R_i(t) = \exp[-\int_0^t \lambda_i(t)dt] \tag{4.30}$$

Since the sub-components are arranged in series, failure of one or more of the N components will cause failure of the whole system. Thus, according to (2.9), the failure probability of the system is given by:

$$P\{S_{sy}\} = P\{X_1 \approx X_2 \approx X_3 \approx \dots \approx X_N\}$$

Or, from (2.10), the reliability of the system requires the reliability of each and every component:

$$R_{sy} = P\{X'_1 \leftrightarrow X'_2 \leftrightarrow X'_3 \leftrightarrow \dots \leftrightarrow X'_N\}$$

where X'_i denote the "non-failure" of X_i .

At this point, we shall assume that component failures are independent events; so the reliability of the system is given by:

$$R_{Sy}(t) = R_1(t) \cdot R_2(t) \cdot R_3(t) \cdot \dots \cdot R_N(t) \quad (4.31)$$

which, upon substituting with (4.30), can be expressed as:

$$\begin{aligned} R_{Sy}(t) &= \exp\left[-\int_0^t \lambda_1(t) dt\right] \cdot \exp\left[-\int_0^t \lambda_2(t) dt\right] \cdot \dots \cdot \exp\left[-\int_0^t \lambda_N(t) dt\right] \\ &= \exp\left[-\int_0^t (\lambda_1 + \lambda_2 + \dots + \lambda_N) dt\right] = \exp\left[-\int_0^t \lambda_{sy}(t) dt\right] \end{aligned} \quad (4.32)$$

where λ_{sy} is the “**system failure rate**” defined by

$$\lambda_{sy}(t) = \sum \lambda_i(t); \quad \text{sum over } i = 1, N \quad (4.33)$$

Discussion: The above in-series model reduces to the “weakest-link” model, discussed in Chapter III, Section III-5, when $X_i = X$ for all $i = 1, N$. In that case, (4.33) yields: $\lambda_{sy}(t) = N\lambda(t)$.

Example 4-14. For a link of N identical elements in series, let the pdf of each element be described by the Weibull function with the parameters m and θ . Then, the corresponding failure rate for each element is given by (4.28):

$$\lambda(t) = (m/\theta)(t/\theta)^{m-1};$$

it follows from (4.33) that

$$\lambda_{sy}(t) = N(m/\theta)(t/\theta)^{m-1} = (m/\theta_N)(t/\theta_N)^{m-1}$$

where

$$\theta_N = \theta/N^{1/m}$$

This result is identical to that given in (3.50). Note: for N identical elements in-series, the failure rate function of the system is also a Weibull function with the parameters m and θ_N ; the parameter m is unchanged, regardless the number of element (N) in the series.

Example 4-15. A computer circuit board is made of 67 components in 16 different categories. The failure rate of each of the 67 components are known from data provided by the various vendors. Assume that all the 67 components are arranged in series; we may then answer a number of relevant questions about the system.

In the table below, the column lists the 16 categories while the second column indicates the number of components (n) in each category; the third column lists the component failure rate λ (constant) and the fourth column is the

cumulative failure rate $n\lambda$ derived from the n components in each category, :

<u>Component type</u>	<u>Number of units, n</u>	<u>Unit failure rate, λ</u>	<u>Cumulative failure rate, $n\lambda$</u>
type-1 capacitor	1	0.0027 x10 ⁻⁶ /hr	0.0027 x10 ⁻⁶ /h
type-2 capacitor	19	0.0025	0.0475
resistor	5	0.0002	0.0010
flip-flop	9	0.4667	4.2003
nand gate	5	0.2456	1.2286
diff. receiver	3	0.2738	0.8214
dual gate	2	0.2107	0.4214
quad gate	7	0.2738	1.9166
hex inverter	5	0.3196	1.5980
shift register	4	0.8847	3.5388
quad buffer	1	0.2738	0.2738
4-bit shifter	1	0.8035	0.8035
± inverter	1	0.3196	0.3196
connector	1	4.3490	4.3490
wiring board	1	1.5870	1.5870
solder connector	1	0.2328	0.2328

Total units: $N = 67$ System failure rate $\lambda_{sy} = 21.672$
x10⁻⁶/hr.

The sum of the second column is 67 as it should; the sum of the last column is $\lambda_{sy}=21.672 \times 10^{-6}$ /hr representing the system failure rate of the circuit board with 67 components arranged in series.

From here, it is straight forward to obtain:

$$R_{sy} = \exp[-21.672 \times 10^{-6} t];$$

and

$$MTTF = 1/\lambda_{sy} = 46142 \text{ hrs.}$$

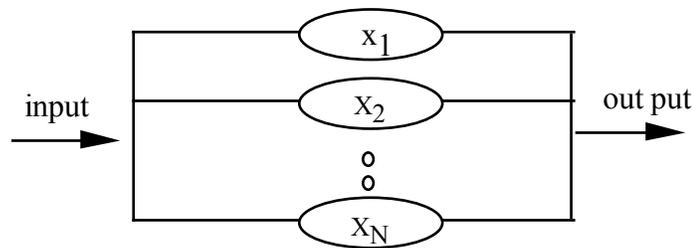
Since λ_{sy} is constant-valued, the time-failure probability of the system is described by the exponential function.

Discussion: To improve the reliability of the circuit board, one may tighten the quality of the flip-flops (9 of them), the shift registers (4 of them), the quad gates (7 of them) and possibly also the hex inverter (5 of them); the failure rate of the connector and the wiring board may also be reduced if

possible.

In general, the overall reliability of a system in-series is not significantly affected by any mutual interaction among its components; in fact, the system reliability is always worse than that of the poorest component. Hence, in practice, one often uses the in-series model to establish a **lower bound reliability** for complex systems whose exact component configurations are not known.

The In-Parallel Models. Suppose that the system S contains N sub-components, denoted by X_i , $i=1,N$; and they are arranged in-parallel as depicted graphically below:



In this case, failure of one component may or may not cause failure of the system. Indeed, this is a system with some "degree of redundancy"; or it is a system built with a "fail-safe" feature. In general, one needs to know the "load sharing mechanism" amongst the components; i.e., the load (or function) carried by one component which fails will be "shared" by the unfailed ones in a certain way (mechanism). In general, failure (or failures) of the unfailed elements will depend on the failure which has just occurred, resulting in a "conditional" situation. Consequently, the related conditional probabilities must be evaluated as a part of the overall analysis.

Given the "load-sharing mechanism", the pdf f_{SY} or the CDF F_{SY} of the in-parallel system can be determined in terms of the component pdf's f_i ; or alternatively, the system failure rate λ_{SY} can be found in terms of the component failure rates λ_i . However, the resulting mathematical complexity in deriving the expression for f_{SY} or λ_{SY} can be excessive even for systems with a simple load-sharing mechanism. The "**bundle theory**" to be discussed later is just such an example.

It occurs in practice that in many an in-parallel system failure of one component does not depend on that of others; and the system can function successfully when at least one component remains functional. This will greatly simplify the complexity of the problem and the reliability function of the in-parallel system is then given by:

$$R_{SY}(t) = 1 - [1-R_1(t)][1-R_2(t)][1-R_3(t)] \cdots [1-R_N(t)] \quad (4.34)$$

Note that the term $[1-R_1(t)][1-R_2(t)][1-R_3(t)]\cdots[1-R_N(t)]$ in (4.34) represents the probability that all N components fail; this will make R_{SY} always better than the best of the R_i 's. Thus, for systems whose component arrangement configuration is not known, one can use (4.34) to establish the **upper bound** for $R_{SY}(t)$.

Example 4-16. Let the reliability of a pressure valve be $R_0=0.8$. With two such valves arranged in parallel, the "system" reliability is given by (4.34) if failure of one valve does not affect the failure of the other:

$$R_{SY} = 1 - (1-0.8)(1-0.8) = 0.96$$

If three valves are arranged in series, the system reliability improves further:

$$R_{SY} = 1 - (1-0.8)(1-0.8)(1-0.8) = 0.992$$

Discussion: parallel structure improves system reliability; but it may be costly.

Example 4-17. Suppose that the valves in the previous example have the time-dependent reliability given by:

$$R(t) = \exp[-\lambda_0 t]$$

Then, the reliability of the system with two valves in parallel is:

$$R_{SY}(t) = 1 - \{1 - \exp[-\lambda_0 t]\}^2 = 2\exp[-\lambda_0 t] - \exp[-2\lambda_0 t]$$

Discussion: If in the previous example N valves are arranged in parallel and it requires **at least** M ($M < N$) of the valves be reliable during a particular application, then the system reliability is determined by the "binomial distribution" (see section II-2 for reference):

$$R_{SY}(t) = \sum_{i=M, \dots, N} C^N_i R(t)^i [1-R(t)]^{N-i} \quad \text{sum over}$$

Series-Parallel Combination Models. Many engineering systems can be configured as a network of several in-series and in-parallel units; each unit may contain components arranged either in-series and/or in-parallel. In general, if the reliability function or the failure rate function of each component in the units is given, one can establish one or more of the following:

(a) consider all components in the network arranged in-series and obtain the **lower bound**

for R_{SY} , using (4.32);

(b) consider all components in the network arranged in-parallel and obtain the **upper bound**

or R_{SY} , using (4.34); and

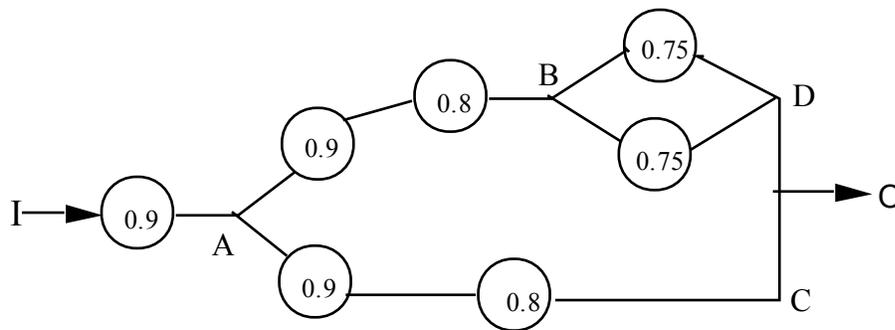
(c) consider the **exact** network configuration and obtain the **exact** R_{SY} , using the method

of **network reduction** technique (to be discussed in Example 4-18 below).

The value

of the exact R_{SY} (c) will fall inside the lower bound (a) and the upper bound (b).

Example 4-18. A system is composed of 7 components arranged in a specific network as shown in the figure below. The individual component reliability values are as indicated in the figure. This system is considered as a combination of in-series and in-parallel units; and it's overall reliability R_{SY} can be evaluated by the above-mentioned procedure, including the **network reduction technique**.



(a) The **lower bound** of the system reliability is given by (4.31):

$$(R_{SY})_{lb} = (0.9)^3 (0.8)^2 (0.75)^2 = 0.26244$$

(b) The **upper bound** is given by (4.34):

$$(R_{SY})_{ub} = 1 - [(1-0.9)^3 (1-0.8)^2 (1-0.75)^2] = 0.9999975$$

(c) The **exact** R_{SY} is determined by the **network reduction** technique:

* the in-parallel unit from B to D is replaced by a single **equivalent** component with

$$R_{BD} = 1 - [(1-0.75)^2] = 0.9375$$

* the in-series unit from A to B to D is replaced by a single **equivalent** component with

$$R_{ABD} = (0.9) (0.8) (0.9375) = 0.675$$

* the in-series unit from A to C is replaced by a single **equivalent** component with

$$R_{AC} = (0.9) (0.8) = 0.72$$

* The in-parallel unit from A to O is replaced by a single **equivalent** component with

$$R_{AO} = 1 - [(1-0.675) (1-0.72)] = 0.909$$

* The overall system reliability is then determined by the in-series unit from I to O:

$$R_{Sy} = (0.9) (0.909) = 0.8181.$$

Discussion: This network has 2 levels of in-parallel units: one is from B to C and the other is from A to O. Such a system is said to have a much greater degree of redundancy than the all-in-series system. The actual reliability in this case is much higher than the **lower bound**; yet, it is also substantially lower than the **upper bound**.

The Bundle Theory. The “**bundle**” theory (due to Daniels, 1945) refers to a bundle of N in-parallel components, where the reliabilities of the surviving components depend on the one which has failed.

Specifically, **Daniels bundle theory** considers a loose bundle of N identical threads. If the bundle carries a total tensile load of (Nx), then the tensile load on each thread is x. When one of the threads fails, the remaining (N-1) threads would share the bundle load (Nx) equally; that is the tension on each thread will be increased to Nx/(n-1). This, in turn, would cause an increase of failure probability for each of the surviving threads. If one or more of the threads fail again, the rest of threads continued to shared the bundle load equally, which of course further enhances the probability of failure of the surviving ones.

Clearly, the ultimate failure of the bundle depends not only on the individual thread's strength but also on the assumed **load-sharing mechanism**. The assumption that the total load on the bundle is always share **equally** by the unfailed threads helps to reduce the complexity of the problem.

Now, let the random variable X be the strength (the tensile load at failure) of the individual threads; Then, the probability that the thread fails at or before $X \leq x$ is denoted by:

$$F(x) = P\{X \leq x\} \quad (4.35)$$

Let the random variable Y_B be the total failure load of the bundle; then $X_B = Y_B/N$ represents the "averaged" thread load based on N -threads. In a way, X is the strength of the thread while X_B is the strength of the bundle. Let the probability that the bundle fails at the strength X_B be denoted by:

$$F_B(x) = P\{X_B \leq x\} \quad (4.36)$$

Note: when X_B reaches x , X reaches x if no thread has failed; X reaches $Nx/(N-1)$ when one of the threads has failed; X reaches $Nx/(N-i)$ when i of the threads have failed.

Thus, given the thread strength $F(x)$, Daniels worked out the bundle strength $F_B(x)$:

$$F_B(x) = \sum_{n=1}^N \sum_r (-1)^{N-n} N! [F(Nx/r_1)]^{r_1} [F(Nx/(r_1+r_2))]^{r_2} \cdot [F(x)]^{r_n} / (r_1! r_2! \cdot \cdot \cdot r_n!) \quad (4.37)$$

where the inner sum is taken over $r = r_1, r_2, \dots, r_n$; and r_1, r_2, \dots, r_n are integers equal or greater than 1; their combination is subject to the condition:

$$\sum_{i=1}^n r_i = N \quad (4.38)$$

For example, for a bundle of 2 threads ($N=2$), the running number n can only be 1 and 2. And, for $n=1$, there can be only $r_1=N=2$; for $n=2$, there can be only $r_1=1$ and $r_2=1$. Accordingly, we obtain from expanding (4.37) the following expression for the CDF of the bundle strength:

$$F_B(x) = 2F(2x)F(x) - F(x)^2 \quad (4.39)$$

Expansion of (4.37) for $N=3$ is similar but is considerably more tedious. Details of which are left in one of the assigned exercises. Beyond $N=3$, the expansion becomes

unmanageable.

When the value of N becomes larger, say $N > 10$, Daniels showed that F_B in (4.37) reduces to the CDF of a normal distribution. In that case, the normal parameters in F_B , namely μ_B and σ_B , can be expressed in terms of the parameters in the pdf of the individual threads (the pdf of the threads need not be normal). That part of the derivation, however, is outside the scope of this Chapter.

Example 4-19. Suppose that the tensile strength X (in GPa) of a single fiber is given by the CDF

$$F(x) = P\{X \leq x\} = 1 - \exp[-(x/8)^7]$$

Then, at 1% of failure probability, the maximum applied fiber stress is determined from

$$F(x) = 0.01 = 1 - \exp[-(x/8)^7]$$

which yields: $x = 4.15$ GPa.

Now, if two such fibers are bundled together, the maximum applied bundle stress at 1% of failure probability is determined via (4.39):

$$F_B(x) = 0.01 = 2[1 - \exp[-(2x/8)^7]][1 - \exp[-(x/8)^7]] - [1 - \exp[-(x/8)^7]]^2$$

which yields: $x \spadesuit 4.02$ GPa.

Discussion: From this example, it is seen that the strength of the loose bundle is actually **weaker** than that of the single fiber. Alternatively, the probability of failure of a bundle of N loose fibers under the load of Nx is actually greater than that of the single fiber under the load of x .

Note: The CDF in (4.39) for the bundle of 2 fibers has been obtained earlier in Chapter III, Example 3.13, case (c). In the latter, the random variable $X=x$ is the applied total load on the bundle; the load on the single fiber is of course $x/2$ when no fiber fails; is x when one fiber fails.

Failure Rate via the Bundle Theory. If the random variable X_B is a time variable, we simply replace x by t in (4.37) to obtain $F_B(t)$; the corresponding failure rate function for the bundle, $\lambda_B(t)$ is then obtained from using (4.1):

$$\lambda_{\mathbf{B}}(t) = [dF_{\mathbf{B}}(t)/dt]/[1-F_{\mathbf{B}}(t)] = f_{\mathbf{B}}(t)/R_{\mathbf{B}}(t) \quad (4.40)$$

Note: though Daniel's bundle theory can become cumbersome when N is large, it is nevertheless useful for a lower-bound estimate for the bundle strength. For tied bundles such as ropes, fiber-reinforced composite materials, etc., the load on each fiber may no longer be assumed to be equally shared.