Engineering EEP 293

Systems - Enhanced Educational Experience for Engineers

Professor Moore

Drexel University

FL 92

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Systems Chapter 1 Study Guide

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B. Introduction

A system is a set of interacting components or elements in which the behavior of each component affects the behavior of the whole set. With this definition, we might describe the ultimate goal of engineering as the design and construction of physical systems to perform given tasks' Therefore, the methods used to analyze and synthesize physical systems are necessary tools for all engineers. The overall objective of this course, therefore, is to introduce you to the standard methods of system analysis and design, and to show that these methods may be applied to all systems, including mechanical, electrical, fluid and pneumatic.

This chapter provides definitions and nomenclature used by system engineers to describe system behavior. These terms will be reoccurring throughout the course.

C. Instructional Objectives

A student mastering this material will be able to

1. Identify a system, its boundary and characteristics.
2. Given system equations or other necessary input-output data,
    determine whether the system is linear or non-linear
    determine whether the system is causal or non-causal
    determine whether the system is time-varying or fixed
    determine whether the system is dynamic or instantaneous
    determine the order of the system.

D. Study Procedure

Read and study Chapter 1. Additional material can be found in references 5, 6, 7, and 13.

Chapter 1
System Definitions

System: A system is a set of interacting components or elements in which the behavior of each component affects the behavior of the whole set. Generally, the systems of interest to us have been constructed to perform some useful task.

System Boundary The boundary is an arbitrary closed line which separates the system from its environment. We usually draw lines to and from the outside of this boundary to indicate interaction between the system and its environment. Outside independent influences which affect the system across this boundary are called inputs or *disturbances* to the system. Inputs are those effects to which the system is designed to respond. For example, the input to the steering system of a car might be defined as the angular position of the steering wheel. The behavior produced in the system by an input is called the system's *response* to the input. Certain parts of the response may be defined as system outputs. The output of the car steering system might be defined as the path of travel of the car. A system disturbance may be defined as an external influence other than the defined input. In the case of the steering system, a disturbance might be a tire blowout or a skid In general, a system can have any number of inputs and outputs. In the figure below, a system boundary is represented by the box. Its inputs (independent variables) are labeled x1 x2, etc., and its outputs (dependent variables) are y l, y2, etc. This is a standard way of representing a system.

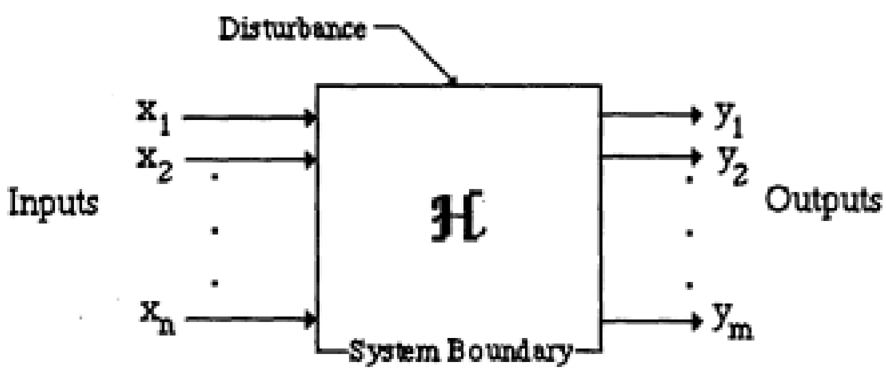


Figure 1 Block Representation of a System

System Characteristic

The H represents the operation that the system performs on the input(s) to produce the output(s). H should be thought of as a mathematical operator and may sometimes be called the system's characteristic, or transfer function.

$$y(t) = H\,[x(t)]$$

Consider the heating system of a house. We might consider it to be a system influenced by the temperature it senses in the house and the thermostat setting (inputs). Its output (assuming a hot water system) might be considered to be a flow of hot water in the radiators. The charactexistics involve the relation between the sensed temperature and thermostat setting which causes the heater to be turned on or off, the speed with which the system responds to changes of inputs, the linkages between the various system components, etc. For a system this complex, a mathematical function H may be difficult to determine directly.

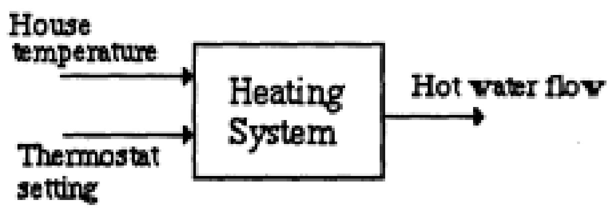


Figure 2 Inputs and Outputs of Home Heating System

Often, large systems can be subdivided into smaller, interacting systems. The heating system box might be replaced with several interconnected boxes representing the thermostat, the heater, the water pump, etc. Each of these could be further subdivided into a number of simpler components. At some point, the detemination of these sub-system transfer functions becomes feasible. It may then be possible to use the interconnection diagram to determine the overall system transfer function.

A large part of this course will be devoted to the determination of the equations relating input and output of various systems, and the solution of these equations to predict system behavior. The methods we will learn do not all apply in all cases to all systems. It is therefore important to recognize certain system properties which may be used to determine the applicable method of solution.

System Classification according to behavior:

Linear / Non-Linear

Most physical systems are non-linear. Unfortunately, most of the methods we use to solve systems apply only to linear systems' We often use these methods anyway and restrict our solution to operating ranges of the system where linearity may be assumed. This is always a question to be considered when modeling large scale systems because ignoring nonlinearities can result in large errors in the solution obtained.

A linear system may be defined as one in which superposition holds between inputs and outputs. Suppose an input x1 produces an output y1 Now suppose we change the input to x2 and observe that the output becomes y2. If the system is linear, the application of (x1 + x2) as an input will result in an output of (y l + y2). That is, the superposition (addition) of the inputs results in the superposition of the outputs that each input produced when it acted alone.

An example of a linear system is the relation in a spring scale between the force exerted on the spring (x = input weight) and its resulting displacement (y = output reading). If the scale registers two pounds when item A is weighed alone, and three pounds when item B is weighed alone, and if it is a linear system, it will read five pounds if items A and B are put on the scale together'

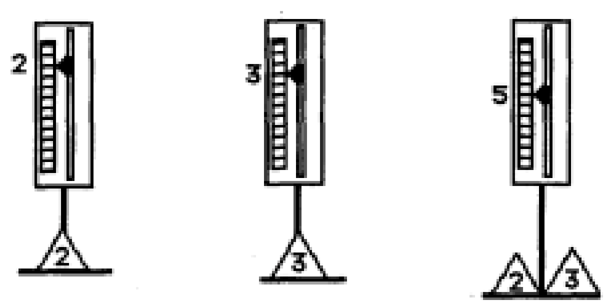


Figure 3 Linear System: Summing inputs produces sum of respective outputs

An example of a non-linear scale would be one which has an upper limit (a mechanical stop) of 4 pounds. Now the indicated weight of the two items together (4 lbs) is not the sum of the weights read individually. This system has a linear range of 0 to 4. The spring scale would also be considered non-linear if we take the output quantity to be energy stored in the spring instead of its displacement. Since energy stored in a spring is proportional to the square of the displacement, d,

$$E = 1/2\ k\, d^2$$

the superposition of two input weights w1 and w2 results in an energy storage of

$$E = 1/2\ k\,(d_1 + d_2)^2$$

where dl and d2 are displacements produced by w1 and w2,respectively. Note that for linearity between input force and output energy, we would need $1/2\ k\,(d_1^{\,2} + d_2^{\,2})$.

<u>Test for Linearity</u>

The basic test for linearity of a system is superposition. This test may be applied experimentally or analytically. For example, suppose we wish to determine whether the system whose differential equation is given by Equation 1 is linear. (Remember that x(t)

$$y\,\frac{dy}{dt} + 4\,y = x\,(t) \tag{1}$$

Assume we have two different forcing functions, x1 and x2, to try on the system. Define y1 to be the response (output) to x1 acting alone, and y2 the response to x2 acting alone. Then the system equation may be written for each case:

$$y_1\,\frac{dy_1}{dt} + 4\,y_1 = x_1 \tag{2}$$

$$y_2\,\frac{dy_2}{dt} + 4\,y_2 = x_2 \tag{3}$$

Now, if the system is linear, then x1 + x2) at the input should produce y1 + y2) at the output. That is, Equation (1) should be true for x = (x 1 + x2) and y = (y 1 + y2) as given in Equation **4.**

$$(y_1 + y_2)\left[\frac{d(y_1+y_2)}{dt}\right] + 4\,(y_1 + y_2) = (x_1 + x_2) \tag{4}$$

To see if this is the case, add equations (2) and (3):

$$y_1\frac{dy_1}{dt} + y_2\frac{dy_2}{dt} + 4(y_1 + y_2) = (x_1 + x_2) \tag{5}$$

This is not the same as Equation (4) because of the product term. Since the superposition of inputs x = (x 1 + x2) does not result in the superposition of outputs y = (y 1 + y2), Equation (1) does not represent a linear system.

In addition to the somewhat more formal test performed above, non-linearity can also be determined from the presence of certain types of terms in the system equation Terms in which either the input or output variables (x or y) or their derivatives appear to a power other than 1, or multiplied together, guarantee non-linearity. Examples are terms such as:

$$x^{\frac{1}{2}} \qquad \left(\frac{dx}{dt}\right)^2 \qquad y\left(\frac{dy}{dt}\right) \qquad x\left(\frac{d^2y}{dt^2}\right)$$

Memory/Memoryless

This designation refers to whether the present output of a system is influenced by past inputs. If our spring scale system had F = ky as its system equation, it would be an "instantaneous" or memoryless system. If the input force suddenly changed from one constant value to another, the output displacement position would instantaneously change with it. We know that since any spring has mass, and because friction may be involved, its motion will not be instantaneous with the applied force, but instead motion will follow changes of force with some delay, and may even overshoot and oscillate about the final position for a time' Actually, the equation for the spring system might be more like

$$F = m\,d^2y/dt^2 + b\,dy/dt + ky \tag{6}$$

where m is the mass of the moving part of the system, b is a friction coefficient, and k is the spring constant. In the figure below, the left panel shows the spring deflection, y(t), for a memoryless system described by F = ky. The panel on the right shows the deflection for a spring system with memory, described by Equation 6.

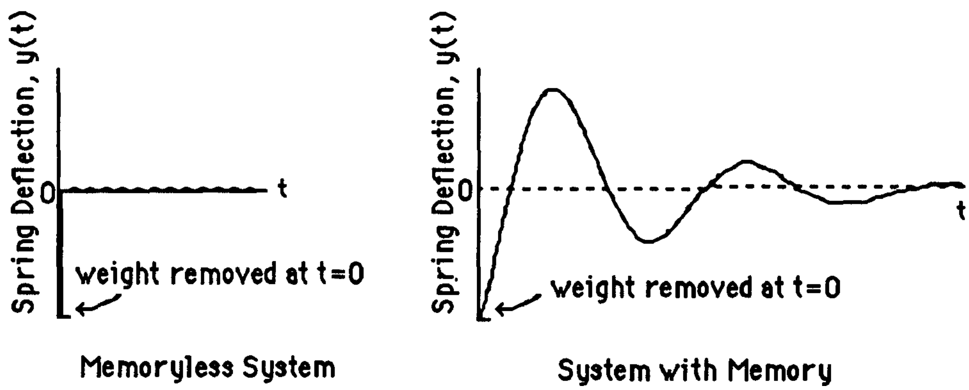


Figure 4 Response of spring system to removal of a weight at t=0

Any system which requires a differential equation to describe its input/output relation has m en more. (Such a system may also be referred to as *dynamic .) To* solve a differential equation, we must integrate over time from the past to the present. Therefore the solution (the present output) depends partially on past history.

The converse of the above rule may not always apply. A system defined by an algebraic equation may also have memory. Consider a tape recorder in which the tape passes first over a record head and then over a playback head. The signal recorded on the tape is x(t). output, y(t), is $y(t) = x(t - \tau)$, where τ is a fixed time delay required for the tape to pass from one head to the next. Since the output is not exclusively a function of the input at the present instant, this svstem has memorv.

Note that the determination of system properties is dependent on a clear definition of system input and output. The present volume of liquid in a tank depends on the entire past history of flow rates in and out of the tank. So if the input is defined as flow in or out, and the output is volume in the tank, the system has memory. Note in the figure below that the output, volume in the tank, may remain non-zero even with a zero input because it "remembers" past flows. However, if we take pressure at the bottom of the tank as the output, and volume of liquid as the input, the system is memoryless.

Algebraic system equations without time delay indicate memoryless

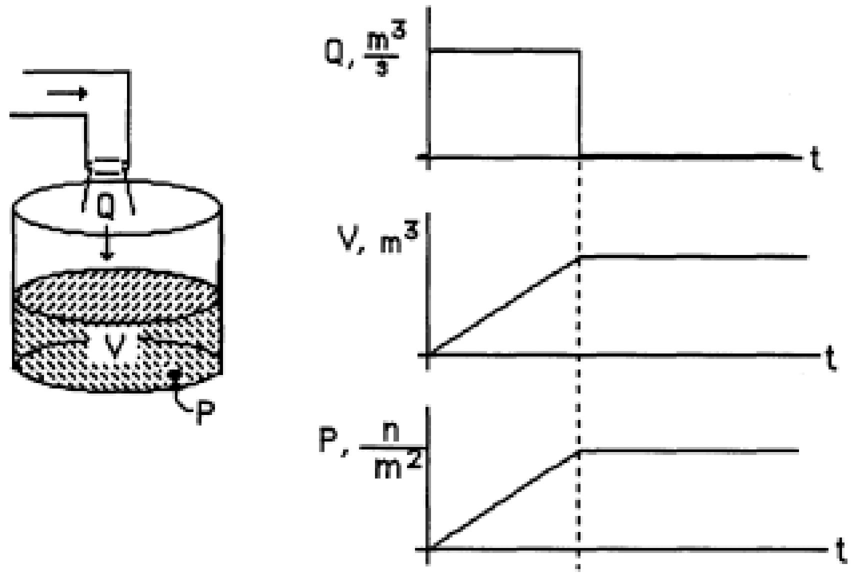


Figure 5 Memory Property Depends on Input and Output Definitions

Time Invariant/Time-varying

In a time-invariant (or fixed) system, the output function is the same regardless of when the input is applied. Suppose x(t) is an input which produces y(t) at the output. Then for a timeinvariant system, if x is delayed by T seconds, the output is also delayed by T seconds, but not changed otherwise. That is, x(t -T) produces y(t -T).

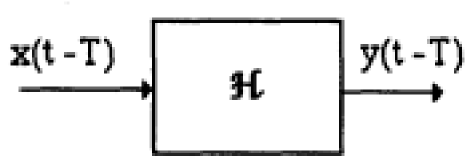


Figure 6 Time Invariant System

If your stereo system is time-invariant, a record sounds the same regardless of when it is played. However, if some components in your amplifier change their value as they heat up, you may note differences in the output between playings of a given record. The system has changed, but not as a function of the input.

Time varying behavior can be noted from the system differential equations. If any coefficient is an **independent** function of time, the system is time varying. For example, the system represented by the equation below is time-varying because of the coefficient of the dy/dt term.

$$4\frac{d^2y}{dt^2} + \cos(2t)\frac{dy}{dt} + 6y = x(t)$$

Note that the coefficient must be an independent function of time. The equation below represents a system which is time invariant, and non-linear.

$$4\frac{d^2y}{dt^2} + y\frac{dy}{dt} + 6y = x(t)$$

Causal/Non-Causal

A causal system is one for which the output does not anticipate the input that caused it. No physical system operating in real time can anticipate an input signal. Such a requirement makes the system non-realizable. An example of a non-causal system equation is given below.

$$y(t) = x(t + 5)$$

In this system, the output, y, is reacting now (t) to an input, x, which will occur 5 seconds from now (t+5). For example, when t = 0, y (0) = x (5). That is, the output at t = 0 reproduces the input which won't be applied to the system until t = 5. In some computer applications, where the input function is a list of values in time order, it is possible to construct a simulation where "future" data points are used to compute the present system output. This is not possible in any real-time svstem.

System Order

System elements we will study can be divided into two types: those which transfer energy into or out of the system (energy sources or sinks), and those which store energy temporarily. In our spring system discussed above, we may consider the mass, friction, and spring elastance as separate system elements. Two are energy storage elements (mass can store kinetic energy, and the spring can store potential energy). The other element acts as a sink for energy (friction converts kinetic energy to heat and this energy is permanently lost from the mechanical system). System order refers to the number of initial energy storage values required to solve for a particular system output. Therefore, the number of indep ndent energy storage elements in a system determines the upper bound on system order. The type of response a system has to any input is a function of its order.

System order can best be determined from the system's differential equation. First, differentiate or integrate through the equation reIating x(t) and y(t) so that no integrals appear (both x and y), and so that a term containing either y(t) or x(t) (undifferentiated) is present. Then system order is indicated by the highest derivative of the dependent variable, y, in the equation. An instantaneous system, y(t) = K x(t), therefore, is of order zero. The system whose equation is y(t) = dx/dt is also of order zero' A zero order system is instantaneous (memoryless).

Because it contains two independent energy storage devices (mass and spring), the spring scale system of Figure 4 can be second order. A look at Equation 6 indicates that if the output is y and the input is F, the system is second order. To be able to solve this equation, the energy stored in each storage element must be known at some specific time' Note that the differential equation which describes the spring system is also of second order (the order of a differential equation is defined as the order of the highest derivative of the dependent variable.). These definitions of order for systems and equations are not exactly the same, since we may differentiate the equation any number of times, but the system order remains two.

In the equations below, y is the system output and x is the system input. System order is specified for each.

second order:

$$\frac{dy}{dt} + 3y = \int x\,dt$$

$$\frac{d^2y}{dt^2} + a\,y = x(t)$$

$$\frac{dy}{dt} + a_1 y + a_2 \int y\,dt = x(t)$$

first order

$$5\frac{dy}{dt} + 3y = x(t)$$

$$y(t) = \int x(t)\,dt$$

$$\frac{dy}{dt} = x(t)$$

zero order (memoryless)

$$y(t) = k\,x(t) \qquad y(t) = \frac{d^n x}{dt^n} \qquad \int y(t)\,dt = x(t) + \frac{dx}{dt}$$

System Classification According to Conservation of Mass or Charge:

To formulate mathematical reIationships which can predict system behavior, we generally make use of two basic laws of physics which apply to any system: conservation of energy and conservation of mass or charge. The state of the mass which is used to store or transfer energy in non-electric systems is often used to further classify the system (although most systems contain elements of more than one classification). We will use the term <u>mechanical</u> system to denote a system in which the energy transfer is among solid bodies. (Mechanical systems will be further subdivided into translational mechanical systems where motion is linear, and rotational mechanical systems where kinetic energy is stored in rotating masses.) If the medium for energy storage or transfer is a liquid, the system is called a <u>hydraul</u>ic system' If the medium is a gas, the system may be called a <u>pneumatic</u> system. The term flui system may refer to either hydraulic or pneumatic systems. Systems for which the energy is stored or transferred by electric charges are called <u>electric</u> systems. A classification which does not exactly fit this framework is that of <u>thermal</u> system in which the storage and flow of heat energy may involve any state of matter.

Chapter 1 Problems

1. Fill in the boxes below with a Y (yes), an N (no), or a number, which answers the questions posed in the left column of the table.

(a) $$\frac{d^2y}{dt^2} + y\frac{dy}{dt} + y = x(t)$$

(b) $$\frac{dy}{dt} + 10y + 5 = x(t+1)$$

(c) $$y(t) = x(t-2)$$

(d) $$\frac{dy}{dt} + e^{-t}y = x(t-3)$$

Property	a	b	c	d
Linear?				
Time invariant?				
Causal?				
Dynamic?				
Order?				

2. Prove that the systems whose input-output equations are given below are non-linear:

(a) $$y(t) + 5 = x(t)$$

(b) $$\left(\frac{dy}{dt}\right)^2 + y = x(t)$$

3. Is the system whose input-output equation is given below causal or non-causal? Show your answer is correct by an example.

$$y(t) = x(t+4)$$

4. For each of the systems whose equations are given below, determine whether the system is linear or non-linear, time varying or fixed. In each case, it is assumed that x(t) is the input and y(t) is the output.

(a) $\frac{dy}{dt} + 10\,y + 6 = x(t)$

(b) $\frac{dy}{dt} + 10\,y = x^2(t)$

(c) $e^{-2t}\frac{dy}{dt} + 10\,y = x(t)$

(d) $y\frac{dy}{dt} + 10\,y = x(t)$

(e) $\frac{dy}{dt} + 10\,y = x(t+3)$

(f) $y\frac{dy}{dt} + 10\,t\,y = x(t)$

5. Consider a car radio as a system. Draw its system boundary showing (and naming) input(s), output(s) and disturbances. Discuss linearity and time variation properties of the system as you have defined it.

6. The equations below represent systems for which x(t) is the input, and y(t) is the output. For each equation, indicate whether the system it represents is linear or non-linear, fixed or time varying, causal or non-causal, and give the system order. (4 answers for each equation)

(a)

$$3\,y + \int_{-\infty}^{t} y\,dt = x(t+2)$$

(b)

$$\frac{d^2y}{dt^2} + y\frac{dy}{dt} = \frac{dx}{dt}$$

(c)

$$\frac{dy}{dt} + t\,y + 4 = x(t-2)$$

(d)

$$\frac{dy}{dt} + \int_{-\infty}^{t} y\,dt = x(t) + 2$$

(e)

$$y(t) = \int x\,dt$$

(f)

$$y\frac{dy}{dt} + 2\,y = e^{-t}\,x(t)$$

7. Use the superposition concept to demonstrate that the system whose equation is y(t) = | x(t) | is non-linear. (The symbols | | denote absolute value.)

Systems Chapter 2 Study Guide

Mechanical System Elements

A. Concepts Addressed By This Chapter

1. Motion, force, energy.
2. Ideal mechanical translational elements
   mass, spring, damper
3. Ideal mechanical rotational elements
   mass, spring, damper
4. Mechanical Transformers, Transducers
5. Modeling of real mechanical system elements.

B. Introduction

This begins our study of a number of idealized elements of various physical systems' Real system components usually possess a combination of the properties we will discuss. For example, in mechanical systems, the analysis of any moving part probably must include consideration of its mass, elastic properties, and friction. The analysis is simpler if these properties are considered to be "lumped" into separate "ideal" devices' An ideal mass element is one which cannot be deformed and which moves without friction. It can store energy (kinetic) which is a function its motion' An ideal spring element is one which has no mass and which can be repeatedly stretched or compressed without any energy loss. It can store energy (potential) which is a function of the reIative displacement of the spring ends. An ideal damper element is one which dissipates energy whenever its terminals experience reIative motion, but it does not store energy. The property possessed by each ideal element is assumed constant. For example, for an ideal mass we rule out changes in mass due to relativistic effects. An ideal spring does not experience changes in its spring constant as might a real spring when it is stretched beyond its normal range. The first step in the analysis of a real system usually consists of modeling real system components with ideal elements.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Identify "across" and "through" mechanical system variables.
2. Write from memory the constitutive reIationship for translational and rotational pure mass, spring, and damper elements.
3..Write from memory the symbols and the elemental relationships between force and velocity for ideal <u>translational</u> mass, spring, and damper elements.
4. Write from memory the symbols and the elemental reIationships between torque and angular velocity for ideal tin mass, spring, and damper elements'
5' Use the elemental reIationships to determine the result of an applied force or velocity to a pure translational element, or an applied torque or angular velocity to a pure rotational element.
6. Calculate work, power, and energy associated with ideal mass, spring, and damper elements of a linear mechanical system.
7 Model a given real mechanical system element by the appropriate ideal element(s). .

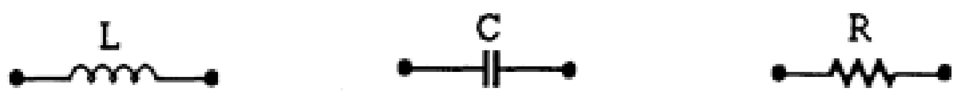
D. Study Procedure

Read Chapter 2. .

Additional material on many of the concepts discussed under this topic can be found in most physics books and in *Vector Mechanics for Engineers* by Beer and Johnston.

E4 Systems
Chapter 2
Mechanical System Elements

In a mechanical system, energy introduced into the system may be stored in moving masses within the system (kinetic energy), or stored in elastic material (springs) within the system (potential energy), or may exit the system at an output port, or may be converted to a form other than mechanical energy (for example, heat). Analogously, in an electric system, electric energy input to the system from, say, a battery, may be stored in an electric field, in a magnetic field, be transferred to a non-electric form, or output from the system in the form of electric energy. Most of the systems we will study will have two types of energy storage and will possess mechanisms for energy to be input and output across the system boundary. To help formulate reIationships for these energy exchanges, it is convenient to replace real system components by connected elements, each of which is related to only one form of energy. Examples with which you may already be familiar are the three elements of the electric circuit:



Inductance accounts for
for
storage of energy in.

Capacitance accounts forResistance accounts
storage of energy. inenergy transferred to a

In mechanical systems, three basic elements may also be defined. Consider a mechanical system in which a massless bumper, backed by a spring, resists an applied force. The spring is initially unstressed at position xO:

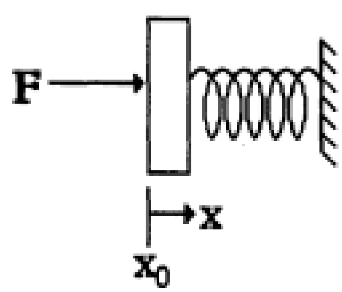
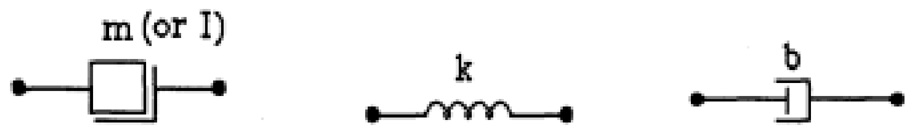


Figure 1 Force Applied to a Mass-Spring System

As the bumper moves to the right, energy is transferred from the force (moving through a distance) to the spring and to its mass. The total energy transferred may be calculated using force times distance. Where does this energy go? Some energy goes to storage as potential energy within the spring. This is a function of the position, x, of the bumper. Since the spring has mass, **there is also kinetic energy present, related** to its velocity. Finally, we must expect that on a molecular level, there will be friction effects within the metal of the spring as it is deformed. This will result in some of the input energy being transformed to heat and lost to the system. The total energy delivered by the force must be equal to the sum of changes in these three quantities: kinetic energy of the spring, potential energy stored in the spring, and (heat) energy transferred or lost.

The symbols defined for mechanical systems to account for these three energy types are:



To account for storage of kinetic energy in a moving mass.

To account for storage of potential energy in an elastic spring.

To account for energy transferred out of mechanical system.

Figure 2 Symbols of Ideal Mechanical Elements

The symbols m (I for a rotational system), k, and b represent values determined from system components such as mass, spring constant, friction coefficient, etc. These will be defined further below.

1. Choice of Variables for Mechanical Translational Systems: Force and Velocity

1.1 Force: A Through Variable

By definition, a system consists of multiple components which operate together to achieve a useful outcome. We may therefore assume that the elements of a mechanical system are mechanically connected, and that one element is able to affect the behavior of another to which it is connected by application of a force at the point of connection. Since this is the basic means of interaction of mechanical elements, we will choose force as a basic variable in our study of mechanical systems'

Consider a series of blocks arranged so that the rightmost block is against a fixed barrier as shown in the figure. A force F to the right is applied to block A.

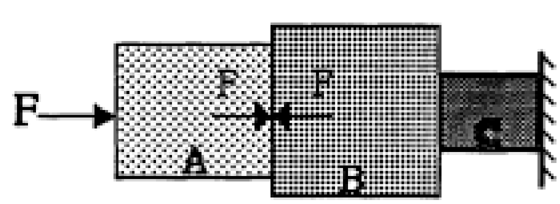


Figure 3 Force Applied to a Series of Blocks

Since block A does not move, the net force on it must be zero. Therefore, it must be receiving an equal but opposite force (-F) from block B. Now, Newton's third law states that the action and reaction forces between bodies in contact must have the same magnitude and line of action, but be opposite in sense. This means A is also exerting a force F on B, to the right. So we might say that the force applied at the left side of A "flows" through A and gets applied to B (and through B to C, etc.). Force is therefore called a "through" variable. A force applied at one terminal of a system element must always flow through the element and exit at the opposite terminal.

1.2 Velocity: An Across Variable

To account for energy in a mechanical system, we need information about velocity of masses (kinetic energy), displacement of springs (potential energy), and relative velocity between damping or friction elements (energy loss). Since displacement and velocity are related, we choose velocity as our second basic variable for mechanical systems.

Velocity is a relative quantity. When we say velocity, we really mean velocity difference between two points. (One point in a system may be defined as the velocity reference point and be assumed at zero velocity, but this choice can often be made arbitrarily) Consider a spring whose endpoints are moving with velocities v1 and v2 with respect to some external reference point.

Figure 4 Spring with velocity difference across its terminals

To determine the energy stored in the spring, we need to know the distance, Δx, the ends have been moved from their equilibrium separation. In terms of velocity, this involves the integral of $(v_2 - v_1)$. So it is the velocity difference across the spring which determines its stored energy. Similarly, the velocity across a damping (friction) element is needed to determine the energy loss involved in its motion. The kinetic energy of a mass depends on the velocity difference between it and its inertial reference frame. Since one terminal of a mass element is always connected to the reference frame, the velocity difference will be referred to as the velocity across the mass element. (We will find in each of our system types that an across variable and a through variable are always necessary for formulation and solution of system equations.)

2. Work - Power - Energy

In a colinear mechanical system (all motion and forces in the x direction), the energy transferred from element A to element B may be calculated as the product of the force A exerts on B times the distance moved. This may be referred to as the work done by A on B. For a small distance, dx,

$$dW=Fdx \tag{1}$$

In the MKS system, work is expressed in newton-meters (joules).

Since we want to use force and velocity as our primary variables, we can substitute for dx:

$$dx = v\ dt \qquad (2)$$

giving

$$\mathbf{dW=Fvdt} \qquad (3)$$

To find the work done by the force on B from time t1 to time t2, we must integrate:

$$W = \int_{t_1}^{t_2} F\ v\ \ dt \qquad (4)$$

Power is defined as the time rate of energy transfer. Mathematically,

$$P = \frac{dW}{dt} \qquad (5)$$

Since the units of W are joules, power is measured in joules per second, or watts. Dividing both sides of Equation 3 by dt gives a formula for the power supplied by force F:

$$P = \frac{dW}{dt} = F\ v \qquad (6)$$

This introduces a relation which will apply to many systems we study:

Power = (across variable) (through variable) (7)

Note that power is an instantaneous quantity That is, power flowing into any element in a mechanical system at any instant in time is equal to the product of the force <u>through</u> that element at that instant, and the velocity <u>across</u> that element at that instant. Unlike the determination of work done, past history is of no significance in the calculation of power. The plot of (Fv) vs. time in Example 1 below is actually a plot of power vs. time.

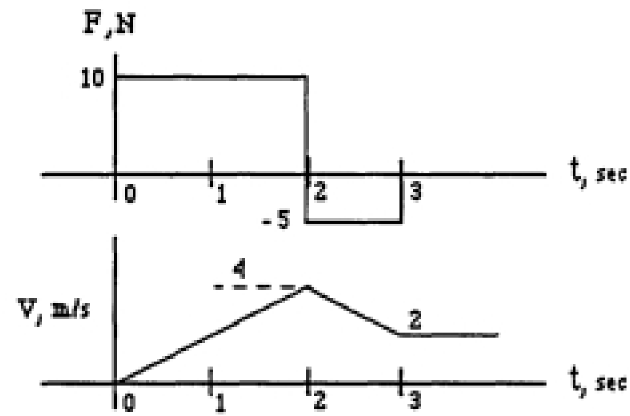
Substitution from Equation 6 into Equation 4,

$$W = \int_{t_1}^{t_2} F\ v\ \ dt = \int_{t_1}^{t_2} P\ \ dt \qquad (g)$$

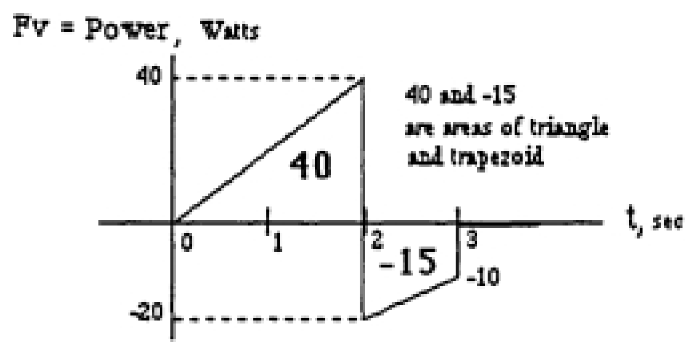
If work is done by a force acting on a mass to accelerate it to some velocity, the mass now possesses some stored kinetic energy. If the mass was originally at rest, the stored energy is equal to the net work done by the force. Although work and energy have the same units (joules), the term work is used only to describe the energy transfer which takes place when a force moves through a distance. Otherwise, the term *energy* should be used. In a mechanical system, we can say that the energy supplied to a system component in any period of time is the sum of all work done on that component in that period.

Example:

Find the total work done by force F acting on an object if the force and the velocity of the object are as indicated by the figures below (both force and velocity are in the x direction).



Solution: We have all the information needed to find work using Equation 4. The force is non-zero only between t = 0 and t = 3, so this is the range over which we must integrate' Since we must integrate the product of F and v, we first plot this quantity:

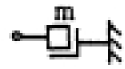
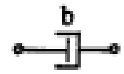


According to Equation 4, work is the area under this curve. Using geometry, the net area after 3 seconds is 25. So 25 joules of work were performed on the object. Note that some area is negative, indicating that energy was transferred back to the driving force during the third second. Total work done is the net sum of all such energy transfers.

3. Basic Translational Elements

As mentioned above, we wish to create models of mechanical systems in which three properties possessed to some degree by all real system components are separately assigned to "pure" system elements. (The term pure means that the element possesses only one of these properties.) The properties and the symbols for the respective elements are given in Table 1. We will now derive reIationships for each of these elements in terms of system variables.

<u>Table 1</u> Elements of the Mechanical System

Property	Element Name	Tie	Energy	Symbol
inertance	mass	A	kinetic	
compliance	spring	T	potential	
friction	damper	D	transfer out of system	

<u>3.1 Translational Mass Element</u>

According to Newton's second law, the vector sum of external forces acting on a body equals the rate of change of momentum of the body.

$$\mathbf{F} = \frac{dp}{dt} \tag{9}$$

By integrating both sides of this equation from $-\infty$ to t we can obtain an expression for momentum. In the equation below, $\mathbf{p}(0)$ is the result of integrating from $-\infty$ to 0. Momentum therefore is an *integrated through variable* .

$$\mathbf{p}(t) = \int_0^t \mathbf{F}\, dt + \mathbf{p}(0) \tag{10}$$

Momentum of an object depends on its velocity according to

$$p = mv \tag{11}$$

The proportionality factor, m, is an inherent property of the object called mass' Equation 11 is the *constitutive* relationship for mass. If m can be considered constant, the mass is called an ideal mass.

To obtain an expression relating across and through variables, substitute from Equation 11 into Equation 9:

$$\mathbf{F} = m\,\frac{dv}{dt} \tag{12}$$

The kinetic energy associated with a pure mass element can be determined by applying Equation (8). To determine the total kinetic energy at time t, we must integrate from the time when the kinetic energy was zero (to).

$$E = \int_{t_0}^{t} F\, v\ dt = \int_{t_0}^{t} \left(m \frac{dv}{dt}\right) v\, dt = m \int_{0}^{v(t)} v\, dv = \frac{1}{2} m\, v^2 \tag{13}$$

We see that the translational mechanical energy stored in a pure mass is a function of its velocity. Since velocity is an across variable, we classify the pure mass element as an **A-type element. Through substitutions from** Equation 10, other expressions are possible for kinetic energy stored in a pure mass:

$$E = \frac{1}{2} m\, v^2 = \frac{1}{2}\ p\, v = \frac{1}{2}\ \frac{p^2}{m} \tag{14}$$

To model the mass property of a system component, we use the symbol below:

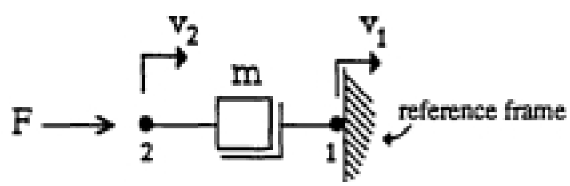


Figure 5 Symbol for Ideal Mass Element

This symbol, like others we will define, has two terminals by which it is connected to the outside world. The through variable (in this case, force) flows in at one terminal, through the element, and out at the other terminal. The across variable (in this case, velocity) is the velocity difference v2 -v l, that is, the velocity across the terminals. For the mass element, the velocity must be measured reIative to the inertial reference frame (usually the earth). Therefore this element must <u>alwas </u>have one terminal connected to this frame (indicated by the shaded region above). NOTE: Failure to observe this rule is one of the most common errors students make! Since the inertial frame terminal is usually considered to be at rest, (vl -0) the velocity across the mass is v2. External elements can exert forces on the mass element only at its free terminal (as indicated by force F in the diagram). The relationship between F and v2 may be expressed in two ways:

$$F = m \frac{dv_2}{dt}$$

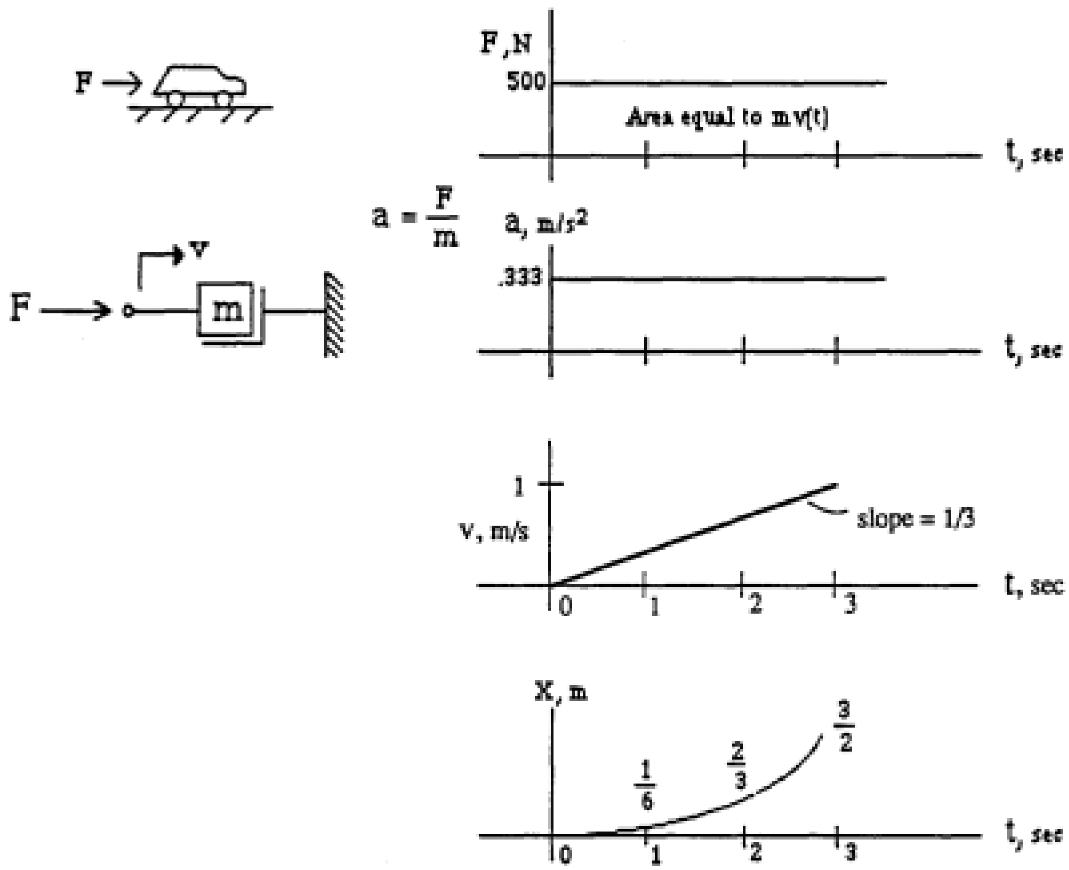
$$v_2 = \frac{1}{m} \int_{0}^{t} F\, dt + v_2\,(0) \tag{15}$$

The latter is a form we will see often in the derivation of system equations for ideal elements. The integral limits are set 0 to t because in many system analysis cases, we are interested in the behavior of a system after a disturbance which occurs at t = 0, or after an input force is applied starting at t = 0. However, the present (time = t) velocity of a mass depends upon the entire past history of forces exerted on it since t = -∞. The evaluation of the integral from t = -∞ to t = 0, yields the velocity at t = 0, v(0), called the initial value or initial condition. This is often a known quantity.

Example 2

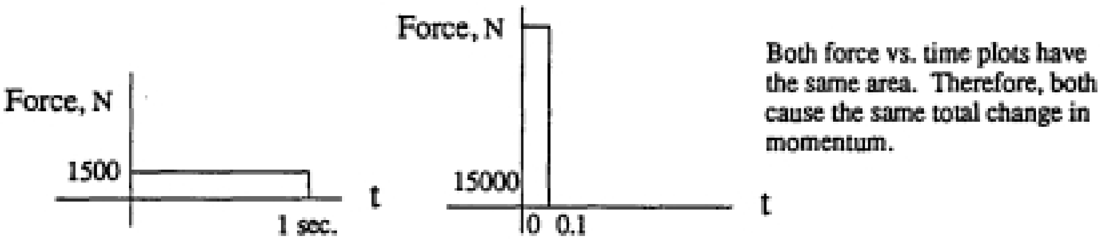
You start (at t = 0) pushing a car of mass 1500 kg from rest, along a level road. You are able to continuously exert 500 N of force on the car. Assuming frictional and other forces can be neglected (not a good assumption in practice) sketch (a) an ideal element representing the car with its terminal values labeled, (b) plots of force applied, acceleration, velocity, and displacement of the car vs. time for t>0. (c) plots of power transfer and kinetic energy of the car vs. time. How much time is required to reach a velocity of 1 m/s?

Solution:



Power is Fv, so the plot for power is the v plot multiplied by 500. Energy is the time integral of Fv, so its plot will have the same parabolic shape as the plot for x but again, multiply by 500.

In Example 2, it took 3s to get the car's velocity up to a 1m/s What kind of force would be required to reach 1 m/s velocity in Is? in 0.1s? Assuming the force is constant, Equation 15 indicates that it is the area under the force vs. time curve which is important' To reach a given velocity, v, from rest, the area must be equal to mv.



Assume a constant force starting at t = 0. Then to reach 1 m/s in 1s we need a force of 1500 N. As we reduce the time to reach a given velocity, the force required increases inversely as the time. To reach lm/s in 0.1s, we need a force of 15,000 N. We can conclude that to achieve an <u>instantaneous</u> change in the velocity of a mass, an infinite force will be required, but only for an infinitesimal time.

<u>3.2 Translational Spring Element</u>

When a solid material is deformed by a compressive or tensile force, it may tend to return to its original dimension when the force is removed' This property will be isolated in our second translational mechanical system pure element, the spring. Again, we consider only the onedimensional case. Suppose a force, F, is applied to one side of an object (drawn as a spring, below) which has this elastic property. The other end is fixed by attachment to the reference frame' We may assume the change in the x dimension of the object is proportional to the applied force with k as the proportionality factor. This is the constitutive relation for the spring element.

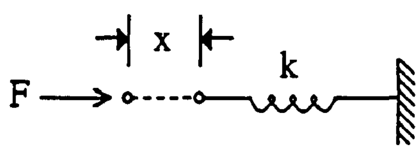


Figure 6 Deflected Spring

$$F=kx \tag{16}$$

Unlike the pure mass element, the spring can have both terminals in motion. Therefore the velocity across the spring is v21 = v2 - v1 and the change in displacement between spring terminals can be obtained from dx = v21 dt. In terms of our preferred across and through variables,

$$F = k\int_0^t v_{21}\, dt + F(0) \tag{17}$$

$$v_{21} = \frac{1}{k}\frac{dF}{dt} \tag{18}$$

Force is a through variable, and therefore, a force applied to one terminal of any pure element must instantly flow through the element to its other terminal where the force is applied to other connected elements' Consider a pure spring element connected to a pure mass element as shown in Figure 7. A force to the right, F, is applied to spring terminal 2. The mass may exert a reaction force, Fs at spring terminal 1. Isolate the spring in a free body diagram.

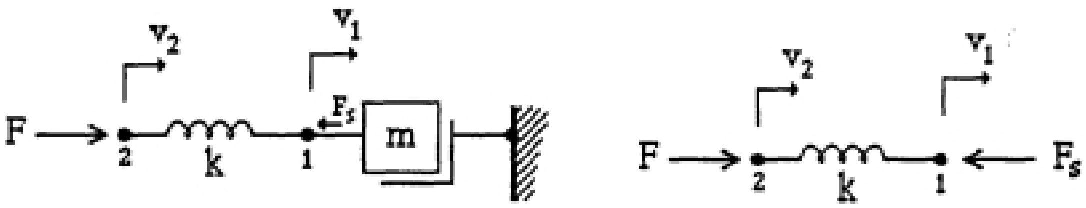


Figure 7 Free Body Diagram of Spring

Since the pure spring element has no mass, an unbalanced force applied to the spring will result in infinite acceleration, including terminal 1. But acceleration of the pure mass element connected at terminal 1 cannot be infinite. Therefore, forces F and Fs must balance, that is F = FS' The spring is exerting a force on the mass equal but opposite to this reaction force. Therefore, the applied force, F, can be pictured as "flowing" through the spring to be applied to the mass. (And then flowing through the mass to be applied to the inertial frame.)

An expression for the energy stored in a spring element can be obtained from Equations 8 and 18. Assume that at t = 0, the spring is unstressed so that stored energy is zero. Then for t>0,

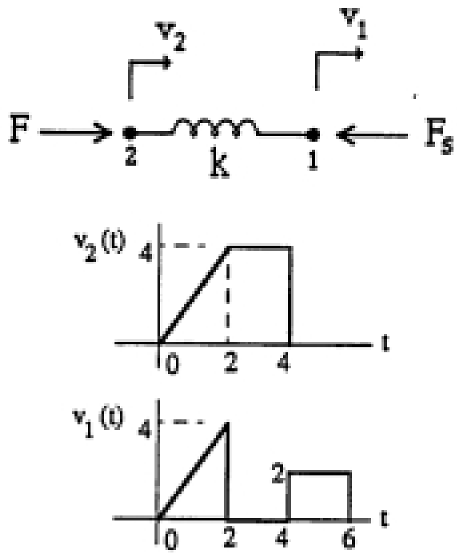
$$E = \int_0^t F\, v_{21}\, dt = \int_0^t \frac{1}{k} F \frac{dF}{dt}\, dt = \frac{1}{k}\int_0^{F(t)} F\, dF = \frac{1}{2k}F^2 \tag{19}$$

We see that the potential mechanical energy stored in a pure spring is a function of the force flowing through the spring. Since force is a through variable, we classify the pure spring element as a **T -type element.** Through substitutions from Equation 16, other expressions are possible for kinetic energy stored in a pure spring:

$$E = \frac{1}{2k}F^2 = \frac{1}{2}x_{21}F = \frac{1}{2}k\,x^2 \tag{20}$$

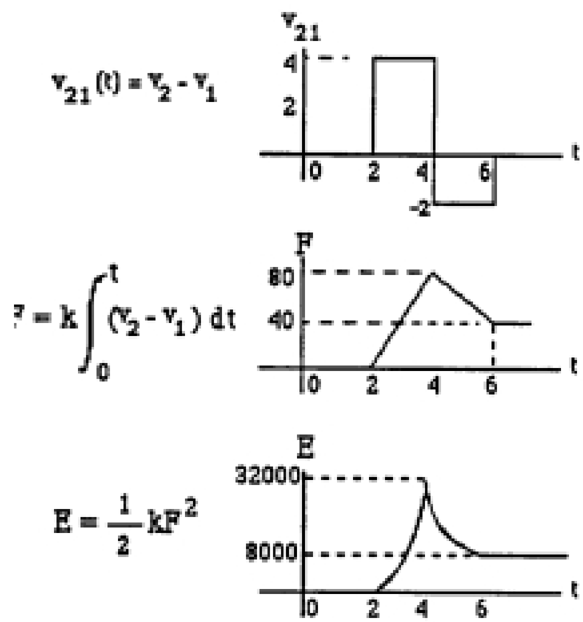
s<u>ample 3</u>

At t = 0, an ideal spring element with k = 10 N/m, has zero force applied to its terminals. The velocity of each of its end points for t > 0 is plotted below. Plot graphs of force, power flow and stored energy vs. time for this spring.



Solution:

Obtain v2 - v1 by graphically subtracting the two velocity plots. F and E can then be found using Equations 12 and 13:



3.3 Translational Damping Element

You are already familiar with a type of sliding friction frequently observed when two dry solid materials slide on each other. The force exerted by one on the other in this case is often assumed proportional to the force normal to the surfaces at the contact point and assumed to be relatively unrelated to the velocity of the sliding. In systems of the type we will study, a more important friction phenomenon is one which occurs when two lubricated surfaces slide together. In this case, the lubricant's viscosity , μ., and the relative velocity, v, of the two sliding surfaces determines the force each exerts on the other. (Viscosity of a fluid may be thought of as a measure of how freely its molecules are able to slide past each other.) For two flat surfaces sliding with a lubricant between them, the force exerted by each one on the other may be

$$F = \mu \frac{A}{d} v \tag{21}$$

where μ is the lubricant's viscosity,
d is the thickness of the lubricating film,
A is the area of sliding contact,
v is relative velocity.

We will combine μ, A, and d into a single factor, b, the damping constant.

$$F= b v \tag{22}$$

We will call an element which has this relation between the force flowing through it and the velocity across its terminals a pure damping element.

Sometimes damping elements are designed to help stabilize mechanical systems. An example is the damping component of a car's shock absorber. Without damping, every bump in the road would produce long term bouncing of the car body. Such dampers may be constructed as a sealed cylinder containing a liquid or gas and a moving piston in which a small hole allows the fluid to pass from one side of the piston to the other.

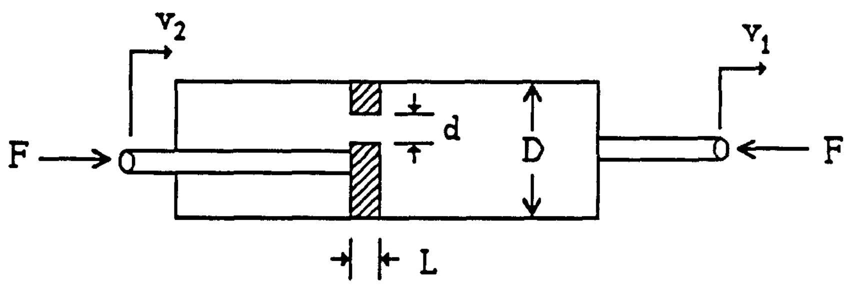


Figure 8 Schematic of Shock Absorber

Again in this case, the force applied and the relative velocity v_{21} between the shock absorber ends are proportional. The formula below can provide a reasonable approximation of the force.

$$F = 8\pi\mu L\left[\left(\frac{D}{d}\right)^2 - 1\right]^2 v_{21} \tag{23}$$

where μ is viscosity of the fluid contained in the cylinder
L, D, and d are dimensions as shown in the figure.

This formula can also be simplified to the form of Equation 22 (F = b v) by collecting all constants in Equation 23 and calling this combined constant b.

The pure damping element we will define for our analysis will represent this kind of friction effect In most cases, b (units are kg/s) is not a constant value except over a small range of velocities. When b can be assumed constant, the element is referred to as an ideal damping element.

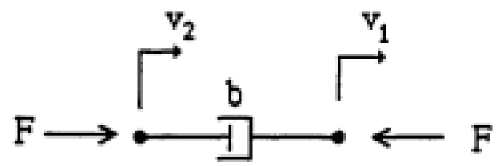


Figure 9 Symbol for Damping Element

$$F = b\, v_{21} \tag{24}$$

$$v_{21} = \frac{1}{b} F \tag{25}$$

From Equation 6, power transferred to any element is the product of through variable F and across variable, v2l. Therefore for the damping element,

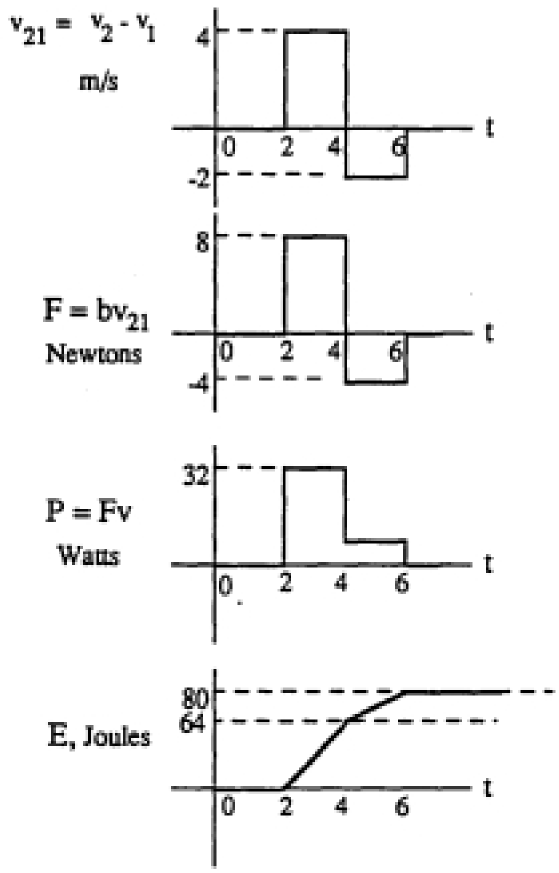
$$P = F\, v = b\, v^2 = \frac{1}{b} F^2 \tag{26}$$

Damping elements absorb input mechanical energy and convert it to heat. We will consider this to be energy lost to the mechanical system Since the damping element <u>dissipates</u> energy, it is designated as a **D-type element.** Energy is not stored in a D-type element. Total energy dissipation over a period of time can be calculated from the integral of power flow to the element.

$$E = \int_{t_1}^{t_2} P \; dt \tag{27}$$

<u>Example 4</u>

The terminals of an ideal damping element of b = 2 have velocities as given for the spring element of Example 3. Plot graphs of force, power and total energy dissipated vs. time for this element.



4. Choice of Variables for Mechanical Rotational Systems:
Torque and Angular Velocity

A rotational mechanical system is one in which the important energy storage and transfer relations of the system elements depend on their rotational parameters, instead of translational forces and velocities. For example, consider the drive shaft of a rear-wheel drive car. At one end, the shaft receives a torque from the engine. The rotating mass of the shaft stores rotational kinetic energy. Some elastic twisting of the shaft may occur under load, thus also storing potential energy. Friction between the shaft and bearings or between parts of the shaft connected by a clutch converts rotational energy into heat. At one end we have torque input from the engine, and at the other, torque is delivered to drive the wheels. We therefore have all the features of the translational system: energy storage, dissipation, and transmission. By choosing torque and angular velocity as our variables, we can derive relationships analogous to those for the translational system.

4.1 Torque: A Through Variable

One way to represent a torque is to picture two equal but oppositely directed (parallel) forces, separated by some distance called the moment arm. Torque is then the product of the force and the moment arm (newton-meters). Analogous to force of the translational system, if a torque is applied at one terminal of a rotational element, it "flows" through the element and is applied in turn to the element connected at the other terminal. Therefore torque is a through variable.

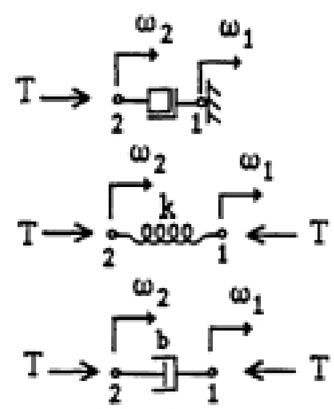
4.2 Angular Velocity: An Across Variable

Energy is stored in a rotational spring by causing its ends to have different angular thereby twisting the spring. Energy is stored in a rotating mass as a function of the angular velocity difference between the rotation of the mass and the reference frame. Energy is dissipated by a rotational damping element such as the clutch mentioned above if one terminal rotates at a different angular velocity than the other. Therefore, each element we define can have an angular velocity difference across its terminals and angular velocity is an "across" variable.

5. **Basic Rotational Elements**

The three rotational elements, inertance, spring, and damper, use the same symbols as their translational analogs. Their constitutive mathematical relationships may be obtained by replacing force and velocity by torque and angular velocity.

property	Element Name		Energy	Symbol
inertance	rotational inertance	A	kinetic	
compliance	rotational spring	T	potential	



5.1 Rotational Mass Element

Newton's second law applied to a rotating mass is

$$\mathbf{T} = \frac{d\mathbf{h}}{dt} \tag{28}$$

where T is torque (N-m) and h is angular momentum (kg-m/s). (Torque and angular momentum are vectors whose direction is determined by the right-hand rule.) To obtain an expression for angular momentum, integrate both sides of Equation 28

$$\mathbf{h} = \int_0^t \mathbf{T}\, dt + \mathbf{h}(0) \tag{29}$$

Therefore, as was linear momentum, h is an integrated through variable. Angular momentum is related to angular velocity, ω, by

$$\mathbf{h} = I\,\omega \tag{30}$$

where the proportionality factor, I, is called the moment of inertia of the rotating object. This is the constitutive relation for rotational inertia elements. (As was the case with translational mass elements, angular velocity must be measured with respect to the inertial reference frame.) Moment of inertia is a function of the distribution of the mass of the rotating object with respect to the axis of rotation:

$$I = \int r^2 \; dm \tag{31}$$

where r is the distance from the axis to mass element dm. An object whose mass is distributed far from the rotational axis (e.g. a bicycle wheel), has a larger moment of inertia than an object of the same mass, but with the mass distributed close to the axis. Combination of Equations 28 and 30 yields

$$T = I \frac{d\omega}{dt} \qquad\qquad \omega = \frac{1}{I} \int_0^t T \, dt + \omega(0) \tag{32}$$

Kinetic energy associated with a rotating mass element can be obtained by application of Equation (7). Since power transfer is the product of the across and through variables,

$$P = (\text{across})(\text{through}) = T\omega \tag{33}$$

Then,

$$E = \int_{t_0}^{t} T\omega \, dt = \int_{t_0}^{t} I \frac{d\omega}{dt} \omega \, dt = \int_0^{\omega(t)} I \, \omega \, d\omega = \frac{1}{2} I \, \omega^2 \tag{34}$$

where t_0 is a time when angular velocity is zero.

Therefore, the rotational mass is an A-type element.

5.2 Rotational Spring Element

The symbol and coefficient used for the rotational spring element are the same as those used for the translational spring:

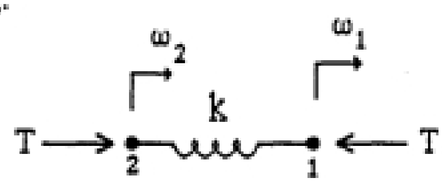


Figure 10 Symbol for Rotational Spring Element

The constitutive relation for the rotational spring is similar to that of the translational element with force and linear displacement replaced by torque and angular displacement:

$$T = k\,\theta_{21} \tag{35}$$

Since θ_{21} is the time integral of ω_{21},

$$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$$

$$T = k\int_0^t \omega_{21}\,dt \;+\; T(0) \tag{36}$$

Since power is P = (across)(through),

$$E = \int_{t_0}^{t} T\,\omega\,dt \;=\; \int_{t_0}^{t} \frac{1}{k}T\frac{dT}{dt}\,dt \;=\; \frac{1}{k}\int_0^{T(t)} T\,dT \;=\; \frac{1}{2k}T^2 \tag{37}$$

Therefore, the rotational spring is a **T -type element.** Other forms of the expression for energy can be obtained from the constitutive relation for a spring element, Equation 35.

$$E = \frac{1}{2k}T^2 \;=\; \frac{1}{2}\theta_{21}T \;=\; \frac{1}{2}k\,\theta_{21}^2 \tag{38}$$

5.3 Rotational Damping Element

Analogous to the *translational case,* elements rotating at different angular velocities and connected by a lubricated interface may exert mutual torques which may be assumed proportional to the relative angular velocities if the elements.

$$T = b\,\omega_{21}$$

$$\omega_{21} = T/b \tag{39}$$

The symbol of a pure rotational damping element is the same as that of the translational element:

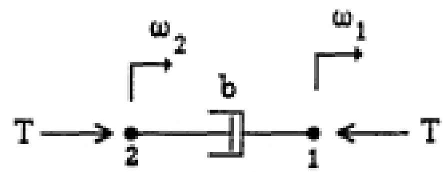


Figure 11 Symbol for Rotational Damping Element

An expression for power dissipation of the rotational damper can be obtained from the (across)(through) relation:

$$P = T\,\omega = b\,\omega^2 = T^2/b \tag{40}$$

6.0 Sources

To this point we have defined what might be called "passive" elements. Passive because mass, spring, damping elements produce no mechanical energy by themselves 'They can only receive energy transferred from other elements and then store, retum, or dissipate it. To model mechanical systems, we need an additional type of element which can be used to model the driving forces of the system. Such "active" elements are called source elements.

A source element is a one-port element which is capable of supplying unlimited energy to a system. An ideal source is a source which maintains a prescribed output under any loading conditions. An *independent* source is one in which the prescribed output is not a function of any other system variable. Sources may either deliver or absorb power. The power associated with a source is the product of its across and through variables (as is the case with any element).

6.1 A-type source

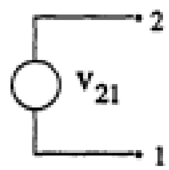
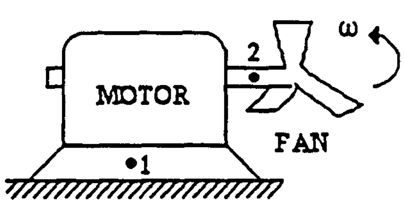


Figure 12 Symbol for Ideal Translational Velocity Source

An A-type ideal mechanical source maintains a prescribed value of velocity difference (either linear or angular, v21 in the above diagram) between its terminals, regardless of other elements connected there. For example, a rotating motor whose speed of rotation is not affected by loading (not attainable with real motors), maintains a fixed angular velocity difference between its output shaft and the reference frame. Therefore, it could be modeled by an <u>ideal velocity source</u>.



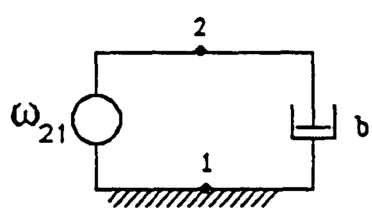


Figure 13 Motor-Fan System. represented by Ideal Velocity Source and Ideal Damping Element

The specified velocity value of an A-type ideal mechanical source is the velocity difference between its terminals. 'Ibis may be specified by referring to numbered terminals (e.g. v21 = -3 means that the velocity at terminal 2 minus the velocity at terminal 1 equals -3), or by using +/signs to indicate that the velocity difference specified is that of the + terminal minus that of the terminal.

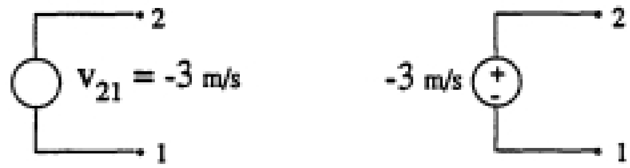


Figure 14 Equivalent Representations of a 3m/s Velocity Source

6.2 T-type source

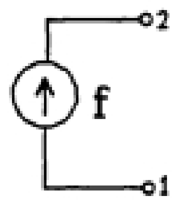


Figure 15 Symbol of Ideal Through Source

An ideal T -type source maintains a prescribed value of the system through variable, regardless of other elements connected at its terminals. The direction of the flow of the through variable is indicated by an arrow on the source. In a mechanical system, a device which can continuously apply a constant force to an object, regardless of the velocity the object attains can be modeled by an ideal T -type source.

7.0 Interconnection of sources and passive elements

The availability of source elements in our system models allows us to "close the loop" between the system and its reference frame. Note that both A-type and T -type sources affect two nodes of the system. A-type sources hold two points at a fixed velocity difference. T -type sources remove a prescribed quantity of the through variable from one system node, and deliver the same amount of through variable at another system node. All passive elements and sources we have discussed are 2-terminal elements (sometimes called one-ports). We assume their mechanical property is lumped between the terminals. The terminals are the only places where we can observe velocity or force associated with the element' The velocity associated with each element is the velocity difference between its terminals. The force acting on each element is the force which flows in one terminal and out the other. The solution of a system's behavior involves choosing the correct set of lumped model elements, connecting their terminals according to the form of the system, writing equations based on the interconnection, and solving these equations.

7.1 Construction of the ideal element model.

The system equations are much more easily written for the lumped element model than for the actual system. Therefore, the first step in the solution is to determine the elements and their interconnection. The best procedure is first to determine how many separate velocities exist in the system. The velocities are the nodes of the system.

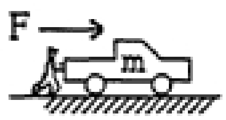
1. Label the velocities of the system and for each velocity draw a node on your diagram
2. Connect the ideal elements representing the properties of each system component between the two nodes which represent the velocities of its terminals.

(In the case of mass/inertia elements, one node must always be connected to the reference frame because kinetic energy must be based on a fixed reference velocity)

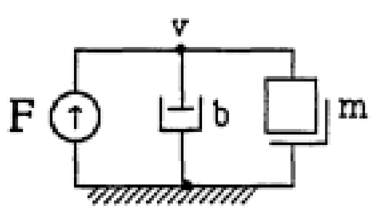
Example:

You are again pushing the car along a level road. This time, we assume the car has mass, m, and that there is friction which can be modeled as viscous friction of coefficient b which your pushing must work against. Draw the lumped, ideal element model of this system.

Solution:



Following the rules set above, we first determine how many separate velocities there are in this system. The answer is one, the velocity of the car (v), which is also the velocity of one if your terminals, your hands (assuming they stay in contact with the car). Therefore, we begin by drawing two nodes, one labeled v and the other the zero velocity node, the ground. Next, we connect all elements indicated by the system between the nodes we have drawn. Any mass element must always have one terminal at reference velocity. The other mass element terminal is the mass' velocity, v. The effect of the viscous friction of the system is proportional to the velocity of the car (with respect to the ground) and so this element also gets connected between these two velocities. Finally, your push is represented by a T -type source. One terminal of this force source clearly goes to the terminal of the mass, v. The other terminal of the force source connects to the ground because this node represents the velocity of your other terminal (your feet, connected to the ground) as you push.

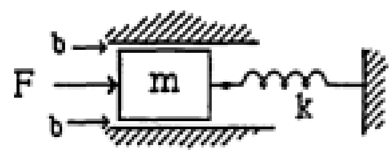


An equation can now be written based on thid diagram. We see that velocity v is across both the ideal mass element and the ideal damping element. According to the force-velocity relations for these elements (Equations 15 and 22), the damping element must be receving a force equal to F = vb, and the mass is receiving a force equal to m dv/dt. Since these forces originate (flow) from the source F, we can write

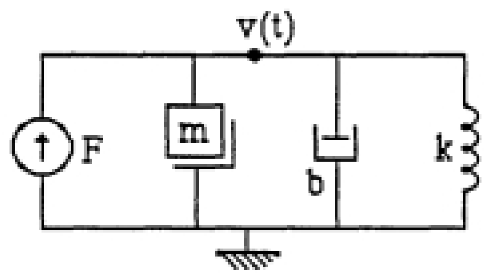
$$F = m\frac{dv}{dt} + b\,v$$

Example:

A force, F, is applied to a mass, m, sliding in a channel with total viscous damping, b, and constrained by a spring of spring constant k, as shown. Draw the lumped equivalent diagram of this system.



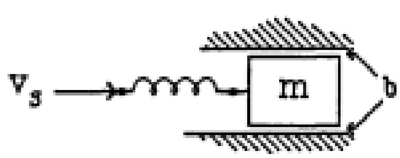
Again, there is only one velocity, that of the mass. Therefore, we sketch one node plus the reference node. Label the node with its velocity, v. The mass is connected from v to reference according to the rule stated above. The spring clearly has one terminal at v velocity (connected to the mass), and the other at zero velocity (connected to the frame). The damping element is the friction between the sliding mass (velocity v) and the frame (velocity zero). So all three passive elements go between the same two nodes. The input is a force applied to the mass (velocity v), so we need a T -type source also connected between the v node and reference. The diagram is:



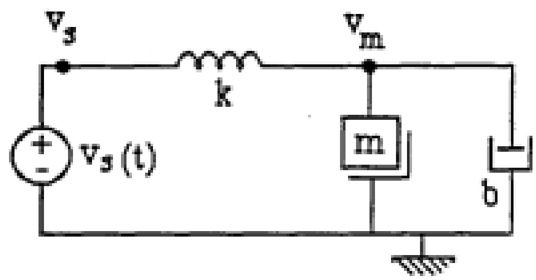
Total force F is divided among the elements according to their

$$F = m\frac{dv}{dt} + b\,v + k\int_{-\infty}^{t} v\,dt$$

Example: The slider system is changed so that the sliding mass, m, is connected by a spring to an ideal velocity driver, delivering velocity vs. Again the total damping between the mass and the frame is represented as b. Draw the ideal lumped element diagram for this system.



This time there are two velocities. The mass velocity, vm, is not the same as the velocity vs, driven by the velocity source. We start the diagram by entering these two velocity nodes plus the reference node. Label one vs and the other vm The mass element (which always connects to ground) must go between vm and reference. The friction is between the mass and the reference frame, so the damping element is also connected between the mass velocity and the frame velocity. The spring has one of its terminals moving at the mass velocity and the other moving at the forcing velocity so it is connected between vm and vs. Finally, the velocity source is connected from vs to ground.



For systems with more than one node or loop, formulation of equations is more difficult. Methods for obtaining equations for systems such as the one above, or of any complexity, will be presented in Chapter 7.

Summary of rules for constructing the ideal element model for mechanical systems:

1. Determine how many separate velocities exist in the system. (Count each mass/inertia component, plus any moving connection points where no mass/inertia element is connected)
2. For each velocity, draw one labeled velocity node. Add a reference node'
3. Connect an ideal mass/inertia element between each of the mass/inertia velocity nodes of step 2 and the reference node.
4. Connect each ideal spring element between the nodes whose velocities represent its end point velocities.
5. Connect each ideal damper element between the nodes whose velocity difference is causing the frictional force.
6. Connect ideal source elements from the reference node to each node receiving an independent force (torque) or velocity input. Assume a positive direction for motion or rotation and set the polarity or direction of each source consistently.

Problems Chapter 2

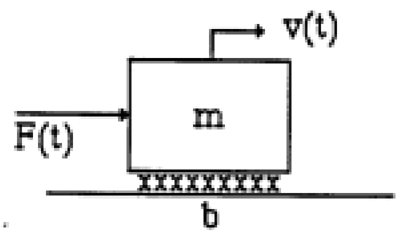
1. Identify each of the following as "across" or "through" type variables:
   (a) translational velocity (b) torque
   (c) force (d) angular velocity

2. Use the constitutive relationship for each ideal element listed below to derive the relationship between its force and velocity.
   (a) ideal translational spring element.
   (b) ideal translational damper element
   (c) ideal translational mass element.

3. Use the results of Problem 2 to derive the equivalent k value of two ideal springs connected in
   (a) series
   (b) parallel.
   (c) repeat the above for two ideal damping elements.

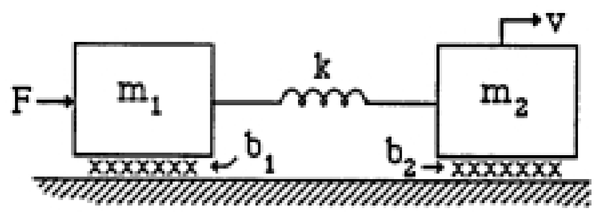
4' Draw an ideal element diagram of each system pictured below. Include source elements'

(a)



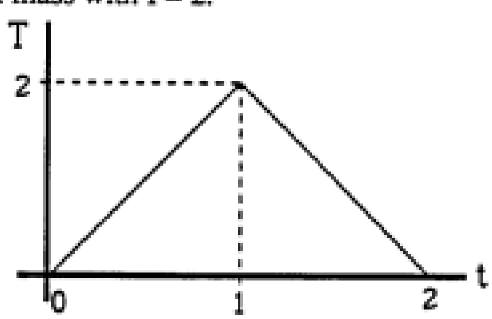
Input F, output v

(b)



5. For the each diagram of Problem 4, what is the order of the system?

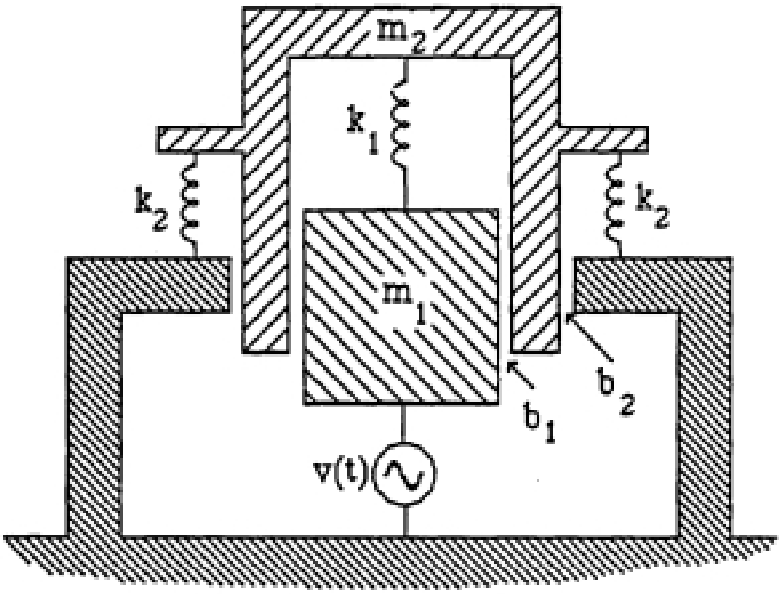
6. The torque T(t), shown in the figure below, is in units of n-m. It is applied separately to each rotational element listed. Sketch the angular velocity, ω, of each as a function of time for
   (a) an ideal rotational damper with b = 2.
   (b) an ideal rotational spring with k = 2.
   (c) an ideal rotational mass with I = 2.



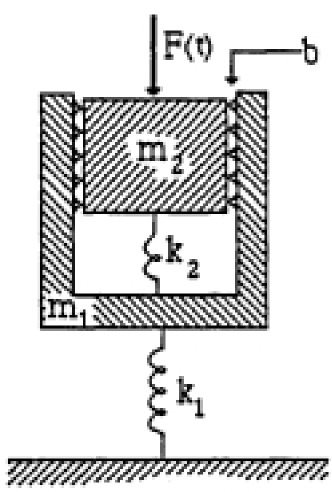
   (d) For each element, find the owe associated with the element at t = 0.5 second.
   Is the flow of power into or out of the element at t = 0.5?
   (e) What energy is stored in each element at t = 3?

7. The system below is a model of an ultrasonic transducer. A motor, represented as a velocity source v(t), drives mass ml. Mass m2 is supported by three springs, two of which (each k2) go to the reference frame, and the other (k1) connects to mass ml. b1 represents the <u>entire</u> viscous friction effect between masses ml and m2. b2 represents the <u>entire</u> viscous friction effect between mass m2 and the reference frame' Motion is possible only in the vertical direction.

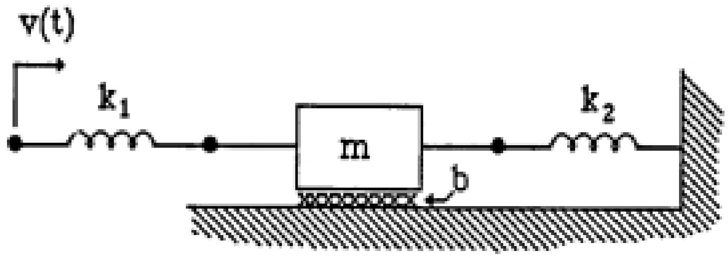
Draw a diagram of this system using ideal elements.



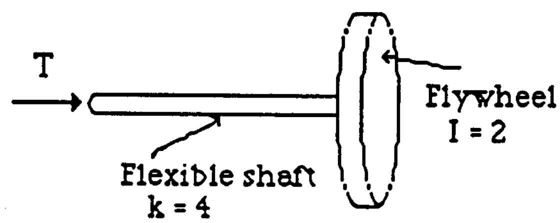
8. Mass m2 is pushed by force F(t) so that it slides into a cup of mass ml, compressing a spring of spring constant, k2. The viscous friction between m2 and the cup has coefficient b. The cup is spring mounted (spring constant k1) to the reference frame. Sketch an equivalent ideal element diagram for this translational mechanical system.



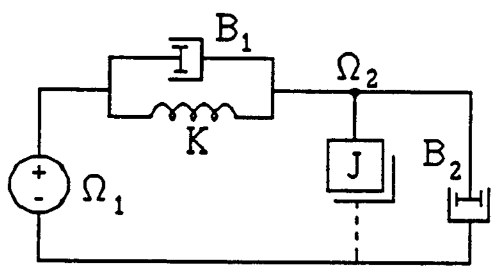
9. In the mechanical translational system shown below, the left end of spring k1 is forced to move at a prescribed velocity, v(t). The other end of this spring is attached to mass m which slides with viscous friction (coefficient b) on the reference frame. The mass is also constrained by spring k2. Sketch a diagram of this system using ideal mass, spring, damper, and source elements.



10. The system below consists of a flexible shaft connected to a rotating flywheel. Assume the shaft can be modeled as an ideal rotational spring of k=4 n-m/r, and that the flywheel is an ideal rotational inertance of I=2 n-m-s^2. Sketch a model of this system using ideal elements.

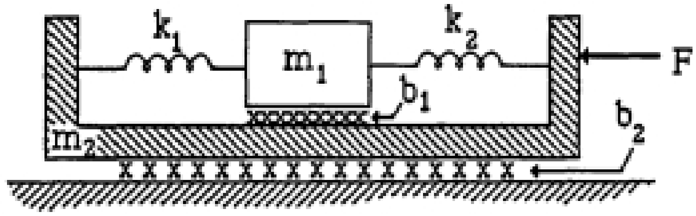


11. For the rotational mechanical system shown below, obtain a differential equation relating rotational velocities Ω_2 and Ω_1. (Arrange with all Ω_1 terms on one side of the equation, and all Ω_2 terms on the other.)

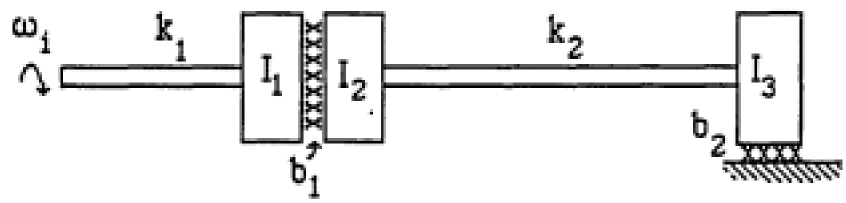


12. Sketch an ideal element model for each of the systems below:

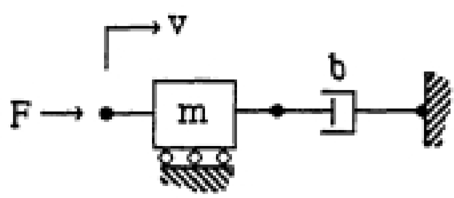
(a) Force F pushes a sled which slides over the ground with viscous friction b2. The mass of the sled is m2. On the sled is a box of mass ml, held in place by two springs as shown. Viscous friction between the box and the sled is b l.

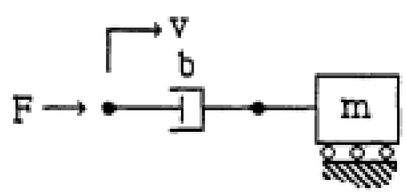


(b) In the transmission system below the input is angular velocity, wi from a motor connected to the left end of flexible shaft k1 This drives heavy clutch plate I, which turns the other plate 12 because of the viscous friction b1 between the plates' Flexible drive shaft k2 then transmits the torque to the load, flywheel 13- 13 rides on bearings which introduce viscous friction b2.



13. Each mechanical translational system pictured below consists of a mass (which slides without friction), and a damping element. Force F is applied to each system as shown. <u>For each case:</u> (a) Sketch a system diagram using ideal lumped system elements and sources. (b) Obtain an equation which relates force F to velocity v.





Systems Chapter 3 Study Guide

Electrical System Elements

A. Concepts Addressed By This Topic

1. Voltage, current, power, energy in electric systems.
2. Ideal electrical system elements: inductor, capacitor, resistor, sources.
3. Modeling of real electrical system elements.

B. Introduction

As was the case with mechanical systems, the analysis of an electric system is made simpler if its components are replaced by "pure" elements. In electric systems, the energy storage devices are the inductor and capacitor, the energy dissipative device is the resistor. As you study these devices, try to note the similarities to the mechanical elements just studied, in their defining equations and in their energy handling properties.

Most real electric components possess more than one of the ideal properties. Again, one task in the analysis process is to determine which properties are important enough to be represented by ideal elements in the equivalent circuit diagram.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Identify "across" and "through" electrical system variables.
2. Write the constitutive relationships for pure inductance, capacitance, and resistance.
3. Write from memory the symbols and the elemental relationships between voltage and current for ideal inductor, capacitor, and resistor elements.
4. Use the elemental relationships to determine the response of an ideal electric element to an applied voltage or current.
5. Given voltages and/or currents, calculate work, power, and energy associated with ideal inductor, capacitor, and resistor elements of a linear electrical system.
6. Replace a given real electrical system element by the appropriate ideal element(s).

D. Study Procedure

Read Chapter 3 of these notes.

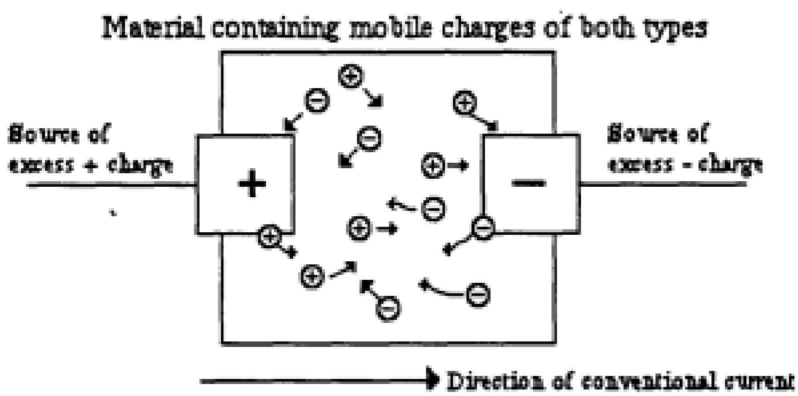
Review Sections 1.0 -1.4 of Principles of Electrical Engineering by Peebles and Giuma, McGraw-Hill, 1991 (MSFE III text).

E4 Systems
Chapter 3
Electrical System Elements

Energy supplied to an electrical system can be accounted for by considering four possibilities' Within the system, the energy may go into storage in an electric field, it may go into storage in a magnetic field, or it may be converted from electrical energy to heat energy. It may also be output from the system to some "load". As was the case with mechanical systems, we will separate these internal energy deposits into three "pure" electric elements. To define these elements and the across-through relationship for each, we must first review our understanding of certain electric circuit quantities and definitions.

1.1 Current: A Through Variable

Charge is a property possessed by certain particles whereby they can exert forces on each other from a distance. There are two kinds of charge, designated by + (charge type of a proton) and -(charge type of an electron). particles of like charge repel each other, particles of unlike charge attract each other. If a material contains charged particles which are relatively free to move (e.g. ions in a solution, electrons in a metal), a flow of these particles (a current) can be produced through the material by creating a charge imbalance at one region in the material and the opposite imbalance at another region. Mobile charges in the material will then move to try to reduce the imbalance. An external source of charge may be used to maintain the imbalance by replacing charges which migrate away from one region and by removing charges which arrive at the other region' In this case, a steady flow of current can be maintained. Since equal current flows at both terminals, we see that current is a through variable.



Current can be a flow of positive charge, negative charge, or both. By convention, the direction of the current flow is taken as the direction of the flow of positive charges. Negative charge flowing in the opposite direction adds to this current' 'The MKS unit of charge is the coulomb, and the symbol for charge is q. Current is measured in coulombs per second or amperes. The symbol for current is i. 'The internationally accepted definition of current and charge is based on the force exerted between parallel current-carrying conductors:

Definition: A current of 1 ampere flowing in each of two parallel wires separated by one meter causes a (repelling) force of $2x10^{-7}$ N/m along the wire

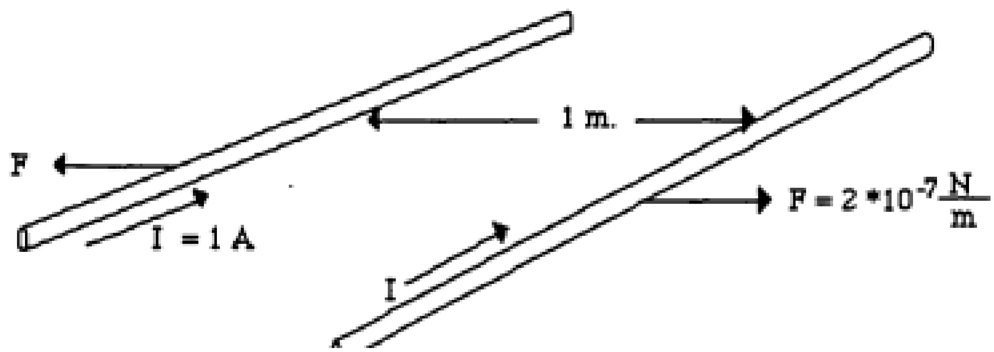


Figure 1 Repulsive force between parallel wires carrying same direction current

An easier definition for our purposes is to define one coulomb as the charge total of $6.24 * 10^{18}$ electrons (or protons). Therefore, in a copper wire, where the primary carrier of current is the electron, a current of 1 amp means that $6.24 * 10^{18}$ electrons per second are passing a given point in the wire.

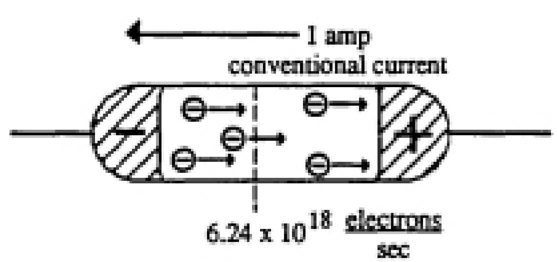


Figure 2 One ampere flowing in a metallic conductor

(Because of the tremendous density of free electrons in a material like copper, the actual average velocity of a given electron is on the order of 0.1 mm/s.)

1.2 Voltage: An Across Variable

Consider a positive charge, Q, fixed at a point in space as shown in Figure 3. A test charge, q, is to be moved from point 1 to point 2.

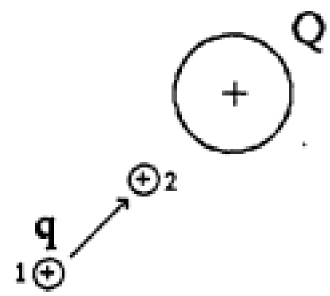


Figure 3 Test charge, q, moved closer to fixed charge, Q

By Coulomb's law, Q exerts a force on q proportional to Qq/r^2, where r is the distance between the charges. We must apply an equal but opposite force on q to cause it to move closer to Q. Therefore work will be done (force times distance) in moving q from position 1 to position 2 in Figure 3. If q is doubled, the force, and therefore the work is doubled. Suppose it requires W joules of work to move q from position 1 to position 2. Then, to move any other charge between these positions would require W/q joules/coulomb. This unit Joules per coulomb) is called the volt. It is a measure of electric potential energy between two locations. The position for which a positive charge has higher energy (position 2, above) is said to be at a higher voltage than other positions. Voltage is always expressed as a voltage difference between two points. Voltage is therefore an across variable.

2. Work - Power - Energy

We have seen that the work done in moving a charge q from point 1 to point 2 is qv21 joules (v21 is the voltage difference between points 2 and 1, that is, v2 - v l). Power, the rate at which work is done is then

$$P = \frac{dW}{dt} = v\,\frac{dq}{dt} = v\ i \qquad \text{watts} \tag{1}$$

Therefore, as was the case with mechanical systems, power is the product of the across and through variables.

Energy stored or dissipated in an electric system may be found from the integral of power.

$$E = \int P\,dt = \int v\,i\,dt \quad \text{joules} \tag{2}$$

Suppose your car's 12 volt battery is used to light a headlamp as shown in Figure 4:

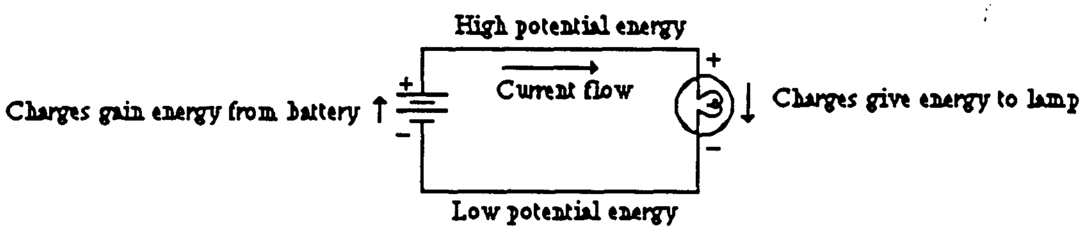


Figure 4 12 volt car battery supplying current to a headlamp

The lamp draws 2 amps of current. The 12 volt battery (through a chemical reaction) gives 12 joules of energy to to each coulomb of charge passing through it. For conventional current this means that + charges enter the bottom terminal at a low potential energy, are raised 12 volts by the battery, and flow out the + terminal with 12 joules per coulomb higher potential energy. 12 joules/coulomb = 12 volts. The power output of the battery (all of which we may assume is consumed by the lamp) is (across) times (through). For electric systems this is voltage times current. The lamp extracts (and converts to heat and light) 12 joules of energy from each coulomb passing through it. Since energy is being taken from the charges, they now enter at the higher potential (+) and exit at the lower potential (-). This must always be the case for an energy dissipative element.

$$p = v\,i = 12\,\frac{\text{joules}}{\text{coulomb}} * 2\,\frac{\text{coulombs}}{\text{second}} = 24\,\frac{\text{joules}}{\text{second}} = 24 \text{ watts}$$

3 Basic Electric Elements

3.1 Capacitor

Consider a battery connected to two long wires as shown in Figure 5' When the switches close, the battery transfers charge to the wires. At some point, the reverse electrostatic force of unbalanced charge on the wires equals the ability of the battery to move more charges onto the wires, and an equilibrium condition results.

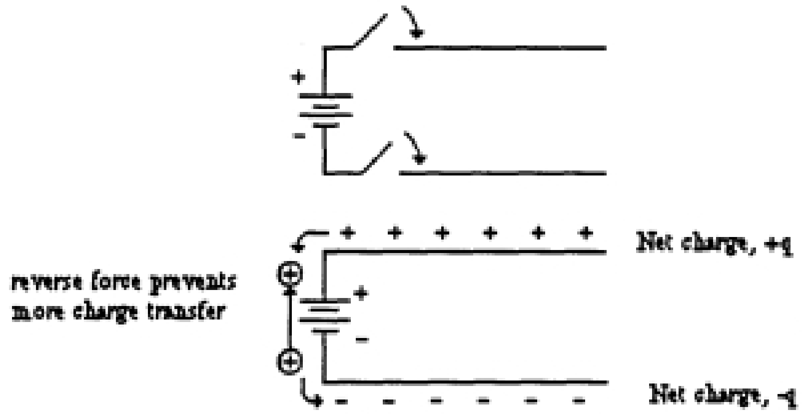


Figure 5 Battery connected to parallel wires.

The amount of charge transferred is determined in part by the voltage of the battery and in part by the structure of the system. An arrangement of two conductors (in the case the two wires) separated by an insulator is called a capacitor. The constitutive relationship of a capacitor is

$$q = C\ v \qquad (3)$$

where q is the charge in coulombs on each conductor v
is the voltage difference between the conductors
C is a proportionality factor called capacitance

The units of C, the capacitance, are coulombs/volt given the name farads. Capacitance is a function of the geometry of the arrangement of the conductors and the insulating material between the conductors. If the wires are brought closer together, the excess negative charge on one wire partially cancels the repelling effect of the excess positive charge on the other wire, thus allowing additional charges to be forced onto the conductors by the battery:

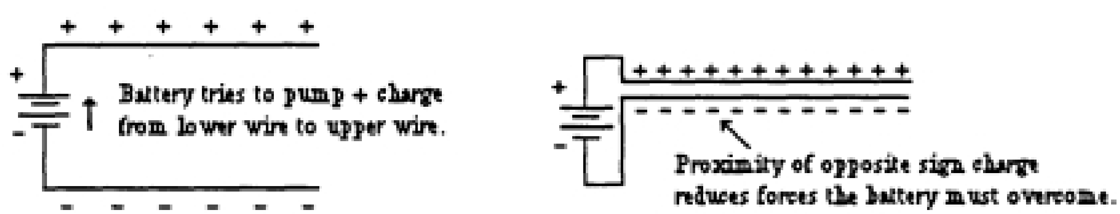


Figure 6 Closer conductors can store more charge

Similarly, if the conductors are metal plates of large area instead of wires, more charge be stored at the same charge density. Therefore, for rectangular plates, we should expect capacitance to vary directly with plate area, but inversly with plate separation.

$$C=\varepsilon A/d \tag{4}$$

where A is plate area
d is plate separation
Є is the permittivity of the material between the conductors

The symbol for pure capacitance element is given in Figure

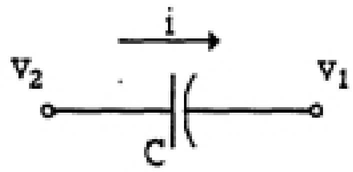


Figure 7 Capacitor Symbol

From q = Cv and i = dq/dt, we have

$$i = C\,\frac{dv_{21}}{dt} \tag{5}$$

where v21 means v2 - v 1, the voltage across the

$$v_{21} = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{0}^{t} i\,dt + v(0) \tag{6}$$

Since q is the time integral of i, it is called the *integrated through variable.*

Since work is done by the battery in forcing charges onto the capacitor plates, a charged capacitor must contain stored energy. The power flow during charging of the capacitor is (across)(through) = v i. Energy stored is the integral of this power, so

$$E = \int_{t_0}^{t} v\,i\,dt = \int_{t_0}^{t} v\,C\frac{dv}{dt}\,dt = C\int_{0}^{v(t)} v\,dv = \frac{1}{2}\,C\,v^2 \qquad (7)$$

where it is assumed that the capacitor was uncharged at t=tO.

We see that a capacitor is an A-type energy storage element because its stored energy is determined by the value of the across variable, voltage. Other formulas for stored energy may be obtained from Equation 3:

$$E = \frac{1}{2}C\,v^2 = \frac{1}{2}q\,v = \frac{1}{2}\,\frac{q^2}{C} \qquad (8)$$

3.2 Inductor

If a magnetic needle is brought close to a wire carrying an electric current, a force of alignment is exerted on the needle. If we place the needle at various positions and observe the direction of forced alignment, we find the needle turns to a position tangent to a circle centered at the wire, in a plane perpendicular to the wire.

To explain this phenomenon, we say that the current sets up a magnetic field in the vicinity of the wire. We visualize the field as directed lines, called lines of magnetic flux. These lines encircle the current according to the right hand rule.

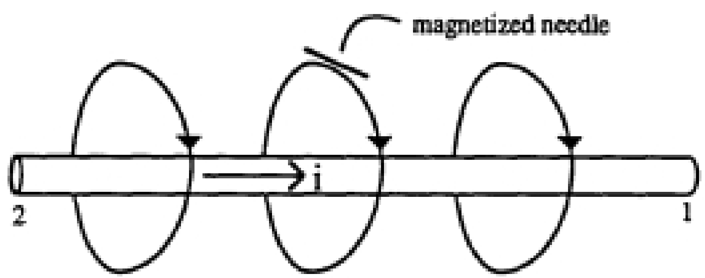


Figure 8 Lines of Magnetic Flux Caused by a Current

The lines of magnetic flux have importance beyond the alignment of a magnetized needle. It has been observed that any change in the magnetic flux linking a wire such as the one pictured in Figure 8, causes a voltage difference to appear (we say the voltage is "induced") across the ends of the wire. The statement of this phenomenon is known as Faraday's Law.

$$v_{21} = \frac{d\lambda}{dt} \qquad (9)$$

Where v_{21} is the voltage difference between points 2 and 1 of the figure, and λ is the flux linkage associated with the wire segment. Flux linkage is the measure of the total encirclements of the conductor by flux lines. In the simple case pictured, flux linkage is 3 (3 lines, each encircling the wire once). Since the flux is produced by the current i, flowing in the wire, flux linkage is proportional to this current

$$\lambda = L\, i \tag{10}$$

The proportionality constant, L, is called *inductance.* Equation 10 is the constitutive equation for inductance. The unit of inductance, flux linkage per unit current, is called the henry. L can be increased by a change in geometry. Suppose the wire is wound as a coil (helix) as shown. The same current will still produce the same 3 flux lines but now, each line links the

So λ has been increased to 6.

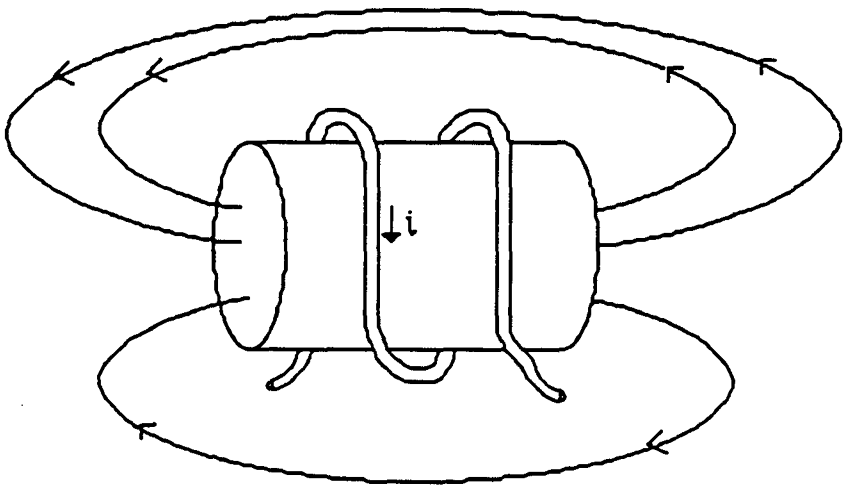


Figure 9 Flux lines of a coil

Inductance has now been doubled since $L = \lambda/i$.

If we substitute from Equation (10) into Equation (9) we get equations relating current and voltage for a pure inductance.

$$v_{21} = L\frac{di}{dt}$$

$$i = \frac{1}{L}\int_{-\infty}^{t} v\, dt = \frac{1}{L}\int_{0}^{t} v\, dt + i(0) \tag{11}$$

Since $Li = \lambda$,

$$\lambda = \int_{-\infty}^{t} v \, dt \qquad (12)$$

X is the integrated across variable.

Equations 11 relate the across and through variables of a pure inductance. The symbol for pure inductance is given below.

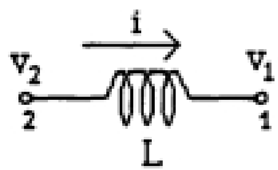


Figure 10 Inductor Symbol

Note from the differential form of Equation 11, that if steady (constant) current flows in the inductor, the voltage across the terminals is zero. Only when the current is changing is a voltage difference induced.

Suppose an inductor's current has been increased from zero to some value, i. During the time the current was changing, a voltage must have appeared across the inductor. Since power is the product of voltage and current, we can determine the total energy stored in the magnetic field of the inductor by integrating the power flow over this time.

$$E = \int_{t_0}^{t} P \, dt = \int_{t_0}^{t} v \, i \, dt = \int_{t_0}^{t} L \frac{di}{dt} i \, dt = L \int_{0}^{i(t)} i \, di = \frac{1}{2} L \, i^2 \qquad (13)$$

Since the energy stored in an inductor is a function of current, a through variable, the inductor is a T-type energy storage element.

By substitution from the constitutive relation, $\lambda = Li$, energy stored in the magnetic field of an inductor can be expressed in other forms:

$$E = \frac{1}{2} L \, i^2 = \frac{1}{2} \lambda \, i = \frac{1}{2} \frac{\lambda}{L}$$

3.3 Resistor ,

When current flows through any material, the material is heated due to inelastic collisions.. of the moving charged particles with the structure of the material. Joule's law (determined experimentally) states that the rate at which electric energy is converted to heat is proportional to the square of the current

$$P = R\, i^2 \tag{15}$$

the proportionality constant, R, is called the resistance of the material. But power is also equal to vi, so

$$v = R\, i \quad \text{and} \quad i = \frac{1}{R}\, v \tag{16}$$

This equation is, of course, known as Ohm's Law, and is the constitutive relation for a resistor. The symbol for a resistor is given below:

Figure 11 Resistor Symbol

The units of resistance, volts/amp or kg m^2 s^{-1} $coul^{-2}$, are defined as ohms for which the symbol Ω is used. It is sometimes convenient to use the reciprocal of resistance, called conductance. Its symbol is G (G=1/R). The unit of conductance (1/ohms) is popularly known as the mho, but the correct MKS system unit is siemens, abbreviated S.

For a cylindrical conductor, resistance is proportional to the length of the cylinder, and inversly proportional to its cross sectional area.

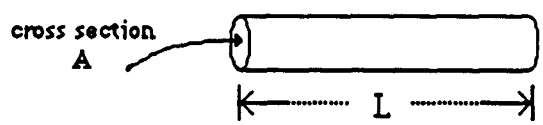


Figure 12 For a cylindrical conductor, resistance is directly proportional to L, inversly proportional to A.

$$R = \frac{\rho L}{A} \tag{17}$$

Where ρ is the *resistivity* of the material. Resistivity is a function of the density of available free charges in the material, their mobility, the temperature, and other factors. ρ has the dimension ohm-meter. Values of ρ for common electric substances can be found in many handbooks. Sometimes it is more convenient to use conductivity, σ, which is the reciprocal of resistivity. The units are mho/meter or siemens/meter.

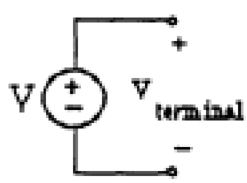
$$\sigma = \frac{1}{\rho} \tag{18}$$

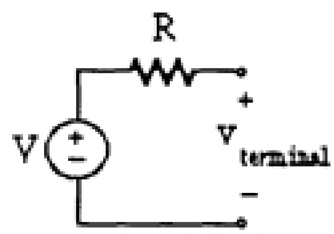
4.0 Electric Source Elements

We repeat the following definitions of sources given in Chapter 2: A source element is a one-port element which is capable of supplying unlimited energy to a system. An ideal source is one which maintains a prescribed output under any loading conditions. An *independent* source is one in which the prescribed output is not a function of any other system variable. Sources may either deliver or absorb power. The power associated with a source is the product of its across and through variables (as is the case with any element).

4.1 A-type electric source: Voltage Source

An ideal voltage source is able to supply any current required to maintain its voltage at the specified value. Consider a 12 volt, ideal, constant source. If we put a 2-ohm resistor across this source, then by Ohm's Law, it will supply 6 amps of current (and 72 watts of power). If we replace the 2 ohm resistor with a 0.2 ohm resistor, the source must deliver 60 amps of current to keep its voltage at 12. It now is delivering 720 watts. An ideal voltage source can adjust to any load without changing its terminal voltage. For real voltage sources, changing the load will usually affect the output voltage. For example, when used to start a car, the 12 volt battery output may fall below 10 volts because of internal losses. This can be modeled as an ideal source in series with a resistor representing the internal resistance of the battery as shown in Figure 13.





ideal voltage source — model of real voltage source

Figure 13 Ideal voltage source (left); model of real voltage source (right)

The symbol for an ideal voltage source is the same as the A-type source of mechanical systems. The use of + - signs to indicate the sense of the assigned voltage value is preferred to the Vab designation.

4.2 T-type electric source: Current Source

An ideal current source is able to supply any voltage required to maintain its prescribed current value, regardless of loading elements connected at its terminals. While most common electrical power sources (batteries, generators) are closer to ideal voltage sources than ideal current sources, there are devices (for example, transistors) which can behave like ideal current sources. Also, it is possible to replace an ideal voltage source and its series resistor with an ideal current source and a parallel resistor as shown in Figure 14. This source transformation between A-type and T-type sources applies to all system types. Note that the symbol for an ideal current source is the same as that of the T-type source of mechanical systems.

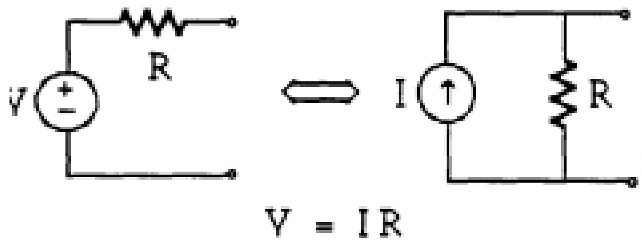


Figure 14. Transformation between source types

4.3 Equivalence of Voltage and Current Source Circuits.

In Figure 15, the same load resistor value, RL, is applied to two real (not ideal) source models which have been constructed to be equivalent according to Figure 14. (The circuits within the dotted lines are models of real sources.) The resulting load voltages and currents delivered to RL by each source are shown. The question is, does v1 = v2, and i1 = i2 ?

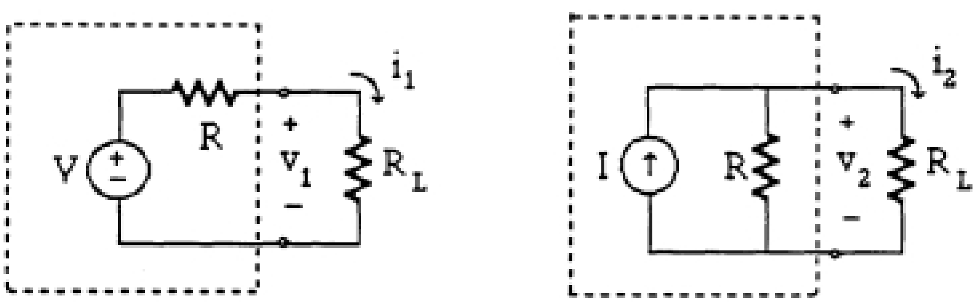


Figure 15 Source transformation preserves voltage and current to load

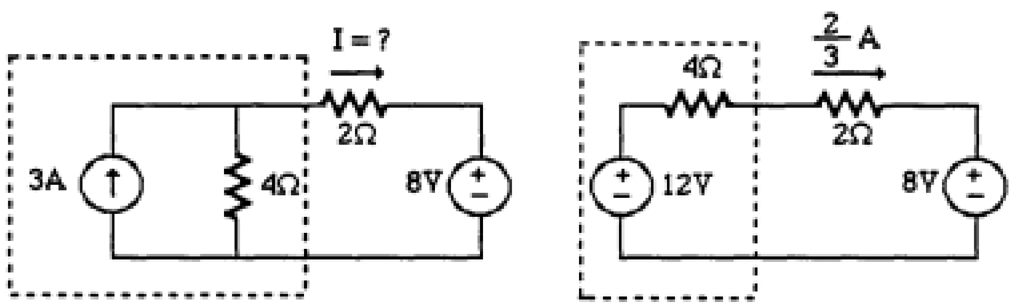
From Figure 15, we see that vi

=V - i1 R

V2 = (I - i2) R = IR - i2R

Comparing these equations term by term, we see that if i1 is to equal i2 and v1 is to equal v2, then the value of the voltage source, V, must equal the value of the current source, I, times R as specified in Figure 14. So either source model may be replaced by the other without affecting the rest of the system. This is often useful in simplifying a circuit before solution. An ideal voltage source in series with a resistor is equivalent to an ideal current source in parallel with a resistor if the resistors are equal (to R) and if V = IR.

Example: Solve for the current in the 2Ω resistor of the circuit on the left:



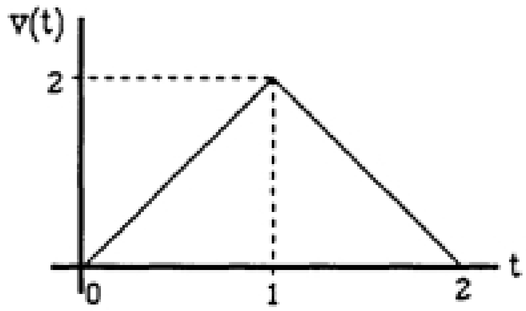
Solution: Convert the 3A current source and its parallel 4Ω resistor to a 12V, 4Ω series combination as shown on the right. The resulting circuit is now a single loop and easier to solve. The current is (12-8)V/(4+2)Ω, or 2/3A.

Chapter 3 Problems

1. (a) Starting with the constitutive relationship for an ideal capacitance (relating charge and voltage), derive the elemental relationship (relating current and voltage).
   (b) Repeat part (a) for an ideal inductance (the constitutive relationship relates flux linkage and current.)
   (c) For both inductor and capacitor, use the fact that the product of the across and through variables gives power to obtain stored energy (as a function of current for the inductor and voltage for the capacitor).

2. The voltage shown in the figure on the next page is applied to each ideal electric element listed below. Find and sketch the current, i(t) vs.time for

   (a) an ideal inductance with $L = 2H$.
   (b) an ideal capacitance with $C = 2F$.
   (c) an ideal resistance with $R = 2\Omega$.

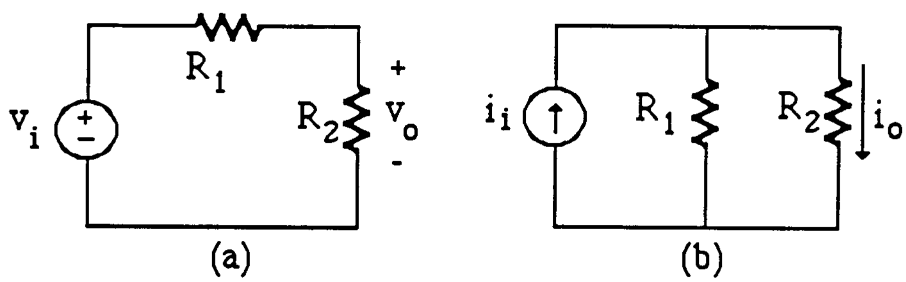


3. For each element of problem 2, find the power associated with the element at t=0.5 second. Is the flow of power into or out of the element at t = 0.5? What energy is stored in each element at t = 3?

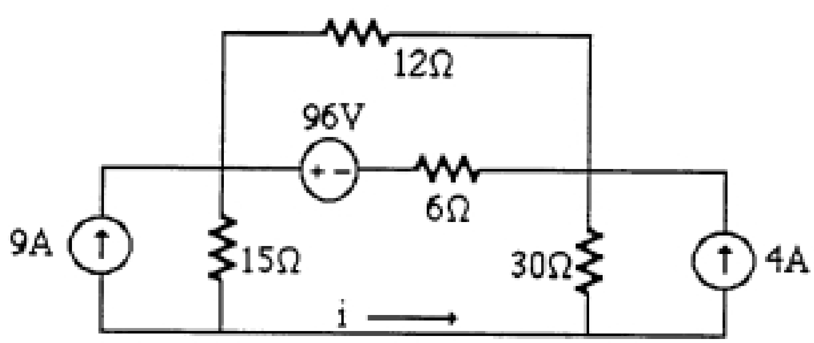
4. An ideal 1 μF (10^{-6} farad) parallel plate capacitor is charged to 10 volts, then disconnected from the charging source.
   (a) What charge difference is now on the capacitor plates? (b) What energy is stored in the capacitor? (c) If one plate is now moved so that the distance between the plates is halved, what will now be the capacitance, charge, voltage, and energy? (d) If the energy values are different in parts (b) and (c), account for this.

5. Using elemental relationships and Kirchhoff's laws, derive the formulas for the equivalent resistance, inductance, and capacitance obtained when two ideal elements (R1 and R2, L1 and L2, C1 and C2) are connected (a) in series, and (b) in parallel.

6. The circuits shown below are known as (a) a voltage divider and (b) a current divider because, in each case, the output quantity (subscript o) is a fraction of the input quantity (subscript i). The fraction is determined by the resistors. For each case, derive the expression for this fraction in terms of R1 and R2.



7. Use source transformations and resistor combinations to reduce the circuit below to a single loop. Solve the loop for the current i, flowing through the bottom wire. (Ans: 1A)



Systems Chapter 4 Study Guide

Fluid and Thermal System Elements

A. Concepts Addressed By This Topic

1. Fluid Pressure, Flow rate, Energy of fluid systems
2. Pure fluid system elements capacitance, inertance, resistance, sources
3. Thermal Temperature, Heat Flow, Energy
4. Pure Thermal Elements
   capacitance, resistance, sources

B. Introduction

Fluid systems can be analyzed using many of the same concepts which we have applied to mechanical and electrical systems. 'The basic across and through variables of fluid systems are pressure (force per unit area) and flow rate (volume per unit time). Fluid systems can store energy in reservoirs (potential energy), in which the stored energy is a function of the volume of fluid stored, or in a moving fluid (kinetic) in which the energy is a function of the flow rate. If fluid flows in a narrow tube or through material which impedes its progress, energy is dissipated from the system. This is an example of a fluid resistance element. The formulas relating pressure and flow for these elements are analogous to those we have seen for the corresponding elements of mechanical and electrical systems.

Thermal systems do not fit our pattern as exactly as the previous systems we have studied. In thermal systems, heat flow and temperature are the through and across variables. Since heat is a form of energy, the rules for energy calculation are different. Heat can be stored and therefore we can define a thermal capacitance element. However, there is no known phenomenon involving heat which suggests a thermal inductance element. Nevertheless, the use of systems methods to solve problems involving heat flow and storage is common in many fields of engineering.

C. Instructional Objectives

A student who masters this material will beableto

1. Identify "across" and "through" fluid and thermal system variables.
2. Write from memory the constitutive relationships for pure fluid inductance, capacitance, and resistance.
3. Write from memory the symbols and the elemental relationships between pressure and flow rate for ideal fluid inertance, capacitance, and resistance elements.
4. Use the elemental relationships to determine the response of a pure fluid system element to an applied pressure or flow.
5. Given pressures and/or flows, calculate work, power, and energy associated with ideal fluid elements.
6. Determine the fluid transformer ratio of a given real fluid transformer and use it to find the transfer of flow, pressure, etc.
7. Model a given real fluid system with ideal elements'
8. Write from memory the relationships between temperature difference and heat flow for ideal thermal capacitance and resistance elements.

D. Study Procedure

Read Chapter 4 of these notes.

Additional material on fluid and thermal systems can be found in references 3, 6, and 7, as well as in *Fundamentals of* Physics, Halliday and Resnick, Third Edition, Chapters 16 and 20.

E4 Systems
Chapter 4
Fluid and Thermal System Elements

A. **Concepts Addressed By This Chapter**

In the previous development of mechanical and electrical system models, we chose convenient across and through variables and defined pure elements which accounted for stored and dissipated energy in the system' We can use our experience with the previous systems to help defme the elements of other system types. In this chapter we consider fluid and thermal systems. For purposes of this course, our discussion of fluid systems will be limited to hydraulic systems in which the medium is an incompressible liquid such as oil or water. The property of incompressibility means that volume is conserved in all system operations. Our analysis of fluid systems will involve calculations regarding the storage and transport of fluid volume. This suggests a similarity to our study of electric systems where charge was always conserved and its transport and storage were calculated' In the case of thermal systems, heat energy is the quantity which must be conserved. The transport and storage of heat is the basis of our thermal systems calculations' Because of these similarities, we will find it convenient to define the fluid and thermal system elements and variables by analogy with electric systems.

B. Fluid Systems

1. Flow: **The Through Variable**

If volume is conserved and is analogous to charge, then the flow variable should be the time derivative of volume, that is,

$$Q = \frac{dV}{dt} \qquad \frac{m^3}{s} \tag{1}$$

where Q is volumetric flow rate in cubic meters per second, analogous to current which is coulombs per second. Volumetric flow rate is a variable related to the macroscopic behavior of the fluid. In the case of liquid flowing through a system, each molecule of the fluid may each have its own velocity, v. Q is much easier to measure and usually is of more significance to system operation. Q is related to the average of all fluid element velocities. In a pipe of cross section A where the average velocity in the axial direction of all fluid particles over the cross section is $\bar{v}$, then

$$Q = \bar{v} A \tag{2}$$

2. Pressure: The Across Variable

Consider a fluid surface acted upon by a force normal to that surface. For example, consider the fluid in contact with the face of a piston to which a force F is applied:

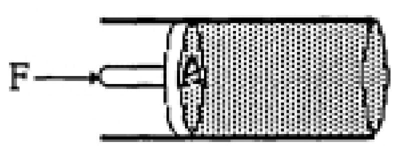


Figure 1 Pressure Applied to a Fluid

The normal force F is spread over the surface of the fluid in contact with the piston. Each surface element, dA receives a small part of the force, dF. The normal force divided by the surface area is defined as pressure, P. Its MKS units are newtons per square meter.

$$P=dF/dA \qquad N/m2 \tag{3}$$

In this case, since we are assuming all dA surfaces receive the same force, pressure is just the ratio of total force to total area:

$$P = \frac{F}{A} \tag{4}$$

3. Work - Power - Energy

Consider a large tank of water with pressure P2 at the entrance of a feed pipe. A piston is used to pump additional water from the pipe of cross sectional area A, where pressure is P1. As force is applied to the piston, the pressure at point 1 rises until it exceeds P2, at which time the valve opens. The piston then moves dx distance against pressure P2, forcing more water into the tank. (Assume the injection of a small amount of additional water into the tank does not change pressure P2.) The volume of fluid moved past the valve into the tank is dV = A dx where A is the cross sectional area of the feed pipe.

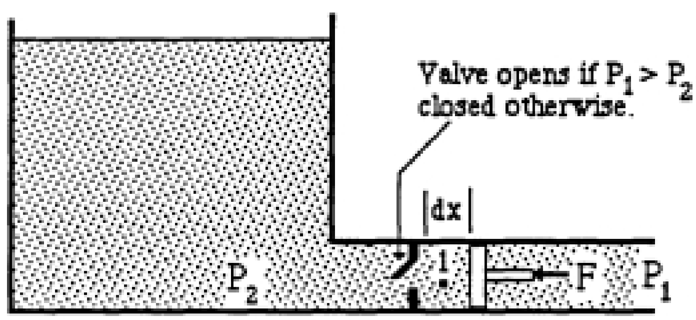


Figure 2 Fluid at point 1 given energy to move past valve to a higher pressure.

The piston force required to overcome the pressure difference is:

$$F=(P2-P1)A =P21\ A \tag{5}$$

The work done by this force in moving the piston through distance dx is:

$$dW=Fdx=P21\ A\ dx=P21\ dV \tag{6}$$

From Equation 6, pressure can be defined as work per unit volume:

$$P_{21} = \frac{dW}{dV} \qquad N/m^2 \quad \text{or} \quad J/m^3 \tag{7}$$

That is, pressure difference can be expressed as joules per cubic meter (as well as the more familiar newtons per square meter). If fluid is moved from a region where pressure is P1 to a region where pressure is P2, each cubic meter of the fluid received (or gave up) P2-P1 joules of energy during the move. Note the analogy between pressure as energy per unit volume and voltage as energy per unit charge.

Power is the time derivative of work:

$$\text{Power} = \frac{dW}{dt} = P_{21}\frac{dV}{dt} = P_{21}\,Q \qquad J/s \quad \text{or watts} \tag{g}$$

Once again, power is the product of the across and through variables.

4.0 Fluid System Elements

4.1 Fluid Capacitance

By analogy with electric systems, fluid capacitance should be an element which can store potential energy (a volume at a pressure). Consider a tank of cross sectional area A, containing fluid to height H, as shown in Figure 3.

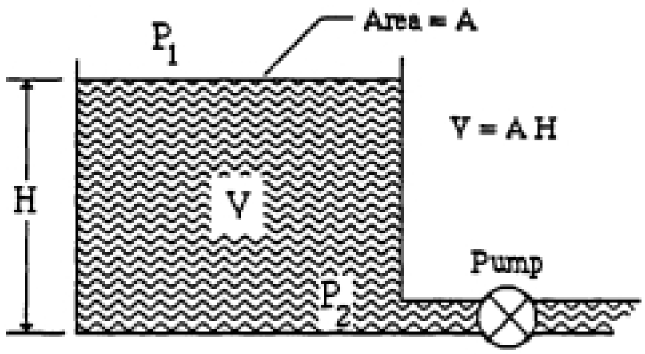


Figure 3 Tank model of fluid capacitance

Pressure at P_2 is ρgH where ρ is the density of the fluid and g is the acceleration of gravity. Volume in the tank is $V = AH$. Substituting for H gives

$$V = \left(\frac{A}{\rho g}\right) P_{21} \qquad \text{or} \qquad V = C\,P_{21} \tag{9}$$

Fluid capacitance, C, is defined as the constant relating pressure and volume. This is analogous to the electric system case where C is the constant relating voltage and charge. Note that fluid capacitance has the dimension volume/pressure which is units of m^5/N or m^4s^2/kg. Equation 9 is the constitutive equation for fluid capacitance.

Other physical structures may be used to obtain the effect of fluid capacitance. Examples shown in Figure 5 include a closed tank with an air space (since air is compressible), or a springloaded piston. An elastic container such as a balloon may also produce this

Figure 5 Other forms of fluid capacitance

The symbol for fluid capacitance is the same as for electric capacitance. A fluid capacitor must always have one terminal connected to the reference pressure node since the pressure of the fluid within the capacitance is measured with respect to this reference.

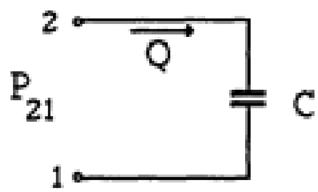


Figure 6 Symbol for fluid capacitance

Since V = C P21, and Q = dV/dt,

$$Q = C\,\frac{dP_{21}}{dt}$$

$$P_{21}(t) = \frac{1}{C}\int_{0}^{t} Q\,dt \;+\; P_{21}(0) \qquad (10)$$

Assume the pressure in a fluid capacitance was zero at sometime, to. Then the energy stored at any time, t, after t = to, can be obtained by integration of power flow (PQ) from to TO t.

$$E = \int_{t_0}^{t} P_{21}\,Q\,dt \;=\; \int_{t_0}^{t} P_{21}\,C\,\frac{dP_{21}}{dt}\,dt \;=\; C\int_{0}^{P_{21}(t)} P_{21}\,dP_{21} \;=\frac{1}{2}\,C\,P_{21}^{2} \qquad (11)$$

Since stored energy is a function of the across variable, P21, fluid capacitance is an A-

Type energy storage element.

4.2 Fluid Inertance

Consider fluid flowing in a cylindrical pipe of length L and cross sectional area A, between two pressures, P2 and PI as shown in Figure 7.

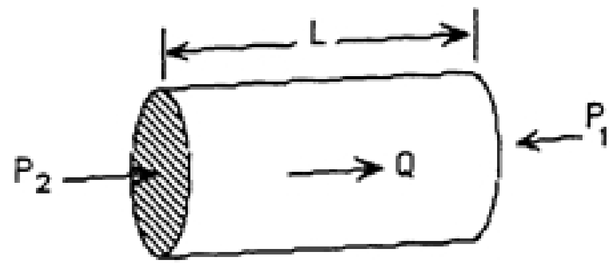


Figure 7 Example of Fluid Entrance

$$F = m\,a$$

$$(P_2 - P_1)\,A = (\rho A L)\frac{dv}{dt}$$

$$\text{Now } v = \frac{Q}{A} \text{ so } P_2 - P_1 = P_{21} = \left(\frac{\rho L}{A}\right)\frac{dQ}{dt} \qquad (12)$$

We define $\rho L/A$ as fluid inertance, L. More generally, fluid inertance is defined as the ratio of pressure difference to rate of change of flow. This is analogous with $v_{21} = L\,di/dt$ for the electric system. The equations relating pressure and flowe in a fluid inertance are given below:

$$P_{21} = L\frac{dQ}{dt}$$

$$Q = \frac{1}{L}\int_0^t P_{21}\,dt + Q(0) \qquad (13)$$

The symbol for fluid inertance is the same as that for electric inductance. The signs of Equations 13 apply if the directions of flow and pressure are chosen as shown in Figure 8.

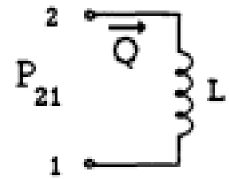


Figure 8 Symbol for Fluid Inertance

Energy storage in moving fluid is a function of the flow rate. Assume that at some time, to, the flow in a fluid inductor was zero. Then the energy stored at any time t, after to can be found by integrating power flow from to to t.

$$E = \int_{t_0}^{t} P_{21}\, Q\, dt = \int_{t_0}^{t} L \frac{dQ}{dt} Q\, dt = \int_{0}^{Q(t)} L\, Q\, dQ = \frac{1}{2} L\, Q^2 \tag{14}$$

We see from Equation 14 that fluid inertance is a T- type energy storage element since the energy stored depends only on the through variableQ

In the electric system, the time integral of voltage was λ, the flux linkage. In fluid systems, the corresponding quantity is the time integral of pressure. This quantity is known as *pressure momentum difference* , Γ

$$\Gamma_{21} = \int_{0}^{t} P_{21}\, dt + \Gamma_{21}(0) \tag{15}$$

To illustrate how Γ is related to momentum, consider a cylindrical mass of fluid with a pressure difference, P_{21}, across its ends. The net x directed force exerted on this mass is $P_{21}A$, where A is the cross sectional area of the cylider.

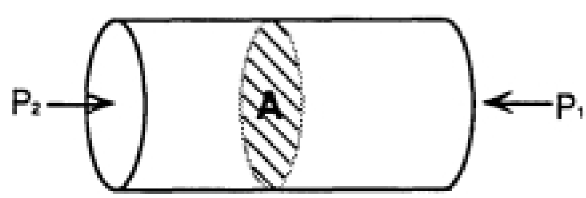


Figure 9 Pressure Momentum difference is the time integral of P2-P1

The pressure momentum difference for this fluid

$$\Gamma_{21} = \int P_{21}\, dt = \frac{1}{A}\int F\, dt = \frac{1}{A}\Delta p \tag{16}$$

where Δp is the change in momentum of the total fluid mass caused by the applied forces.

From Equation 16,

$$P_{21} = \frac{d\Gamma_{21}}{dt} \tag{17}$$

Analogous to the electric case, this indicates that a pressure difference is "induced" across the element whenever there is a change in pressure momentum' From Equation 13,

$$\frac{d\Gamma_{21}}{dt} = L\,\frac{dQ}{dt} \tag{18}$$

which leads to the constitutive relation for fluid inertance,

$$\Gamma_{21} = L\,Q \tag{19}$$

4.3 Fluid Resistance

Viscosity

Before deriving expressions for the D-type fluid system element, we will discuss a property of any fluid which is the basis of fluid resistance, viscosity. Consider a small rectangular surface element of a fluid. The element has surface area A and thickness d. A force acts on the surface at some angle:

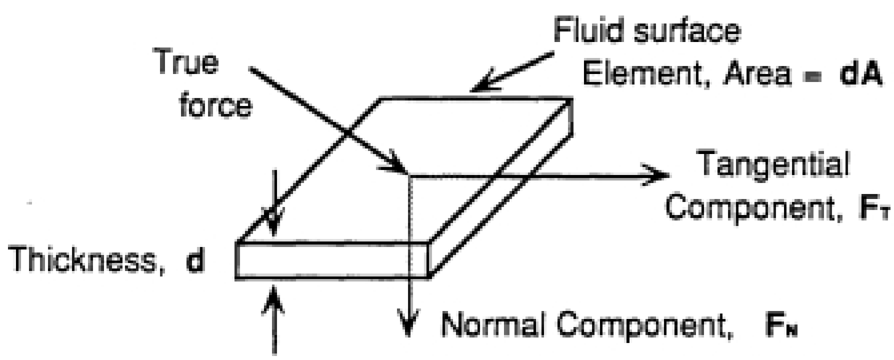


Figure 10 Components of Force on Fluid Element

We have defined the normal component of force, divided by the area, as pressure. The tangential component of force tends to slide this surface element over its neighbor. Tangential force divided by the area (F_t /A) is called "shear stress". Velocity gradient is the change in x direction velocity per unit thickness observed as we travel from the top to the bottom face (y direction). For a very thin element it may be considered to be the velocity difference between the top and bottom faces of the element, divided by its thickness, d. In Figure 11, the top and bottom faces of a thin element of fluid are acted upon unequally by sliding neighbor elements so that the tangential velocities of these surfaces differ.

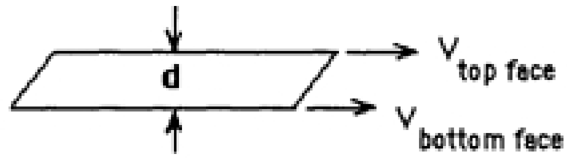


Figure 11 Velocity Gradient Across a Fluid Element is $(V_{top} - V_{bottom})/d$

The velocity gradient is a measure of the rate at which the element is deforming. Such deformation is called strain. The ratio of shear stress to the resulting rate of fluid deformation is called viscosity, μ.

$$\mu = \frac{F_t /A}{\text{velocity gradient}} \qquad \frac{Ns}{m^2} \tag{20}$$

It may be easier to think of viscosity as a measure of the force required to produce a given relative velocity between two objects in sliding contact with a lubricant between their faces (the lubricant being the thin fluid element). Suppose a metal cube, C, is pushed at constant velocity across a metal plate on which there is a film of lubricating oil:

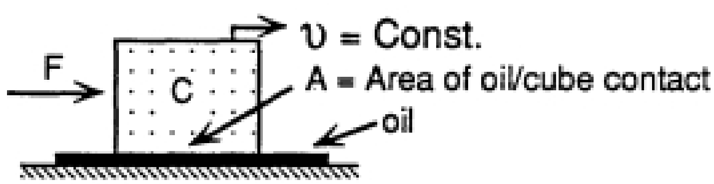


Figure 12 Block Sliding on Oil Film

We assume the upper surface of the lubricant moves with velocity C. The lower surface remains at zero velocity, in contact with the plate. The viscosity of the lubricant determines the velocity of the cube since

$$\mu = \frac{F/A}{\left(\frac{v_c - v_p}{d}\right)} = \frac{\text{shear stress}}{\text{velocity gradient}} \tag{21}$$

Viscosity is a measure of the "thickness" of the fluid. Liquids which flow slowly, such as molasses or heavy oil have high viscosity. Fluids such as water or alcohol have lower viscosity.
In the example above, force is being applied to move the cube through a distance; therefore

work is being expended as the cube moves at constant velocity. This energy is being dissipated as heat in the sliding molecules of the lubricant. Viscosity is therefore a measure of frictional effects within a fluid.

When fluid is forced to flow through narrow tubes or small openings, frictional losses cause energy to be removed from the fluid system and converted to heat. The resulting pressure drop is related to flow rate. We define fluid *resistance,* R, as the proportionality factor relating pressure to flow:

$$P21 = R\ Q \tag{22}$$

This equation is analogous to the Ohm's law equation of electric systems. The symbol for fluid resistance and the assumed directions of flow and pressure are given below.

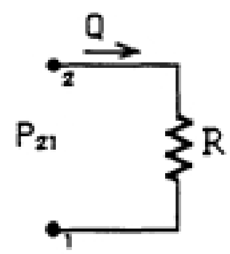


Figure 13 Symbol for Fluid Resistance

Power flow into a fluid resistance element is obtained as usual from the product of the across and through variables That is,

$$\text{power dissipation} = P_{21}\,Q = R\,Q^2 = (P_{21})^2/R \quad \text{watts}$$

Since all power to a fluid resistance element is dissipated as heat, a fluid resistance is a DType element

Fluid resistance in general is not a linear, constant quantity over all flow rates. In some cases, a value for R can be found which works well over a limited range of flow rates. For incompressible flow through a long capillary tube, the Hagen-Poiseuille Law for laminar flow gives R as

$$R = \frac{128\mu L}{\pi\, d^4} \tag{23}$$

where μ is viscosity, L is tube length, d is inside diameter

Of course, resistance to fluid flow can be introduced in a fluid system in other ways, including porous media, baffles, and other flow restrictions. In most cases, these are represented by a relation as simple as Equation 22 only over a limited range of flow rates.

C. Thermal Systems

One of the forms of energy we have encountered in each of the preceding systems is heat. Heat is generated in the operation of almost any type of dynamic system. In the systems studied so far, this heat was developed in the D-type elements of the system. But what becomes of the heat after it is generated? We define thermal system elements in order to study heat flow and temperature distribution in materials. For example, electronic devices such as transistors or integrated circuits develop heat as they operate. If this heat is not removed, high operating temperatures can cause improper operation or damage the device. Automobile engines operate best after they "warm up" to temperatures too hot to touch. Yet, as you may have experienced, if heat is not constantly removed by the engine's cooling system, it will not operate for long. The furnace that heats your home can operate much more inexpensively if you improve the insulation in the walls of your house by something called an "R" factor, a measure of the ability of the insulation to prevent heat flow from the air inside the house to the outside. Since friction and energy losses in the form of heat are part of virtually any system, we need a method to account for heat generation, flow, and loss.

1. Temperature: An Across Variable

By analogy with previous system analysis methods, we know we need an across variable and a through variable as a basis of our thermal system analysis. The across variable for thermal systems is temperature, T. (We will use the letter T because it is easy to remember, confusion with torque should not be a problem.) Temperature is actually a measure of the kinetic energy of the molecules of a substance. If a material of a high temperature (fast moving molecules) is brought in contact with a material of a lower temperature (slower moving molecules), molecular collisions between the two will, in time, bring their average molecular Kinetic energies (and thus their temperatures) to equal values. Temperature is measured on an arbitrary scale based on the properties of water' Degrees kelvin, Celsius or Fahrenheit are non-dimensional units.

2. Heat Flow: A Through Variable

Heat is a form of energy. However, heat is often expressed in units of calories instead of the preferred energy unit, joules. The calorie was originally defined as the amount of heat required to raise a gram of water one degree C. However, the heat required for this varies somewhat over the temperature scale. To avoid ambiguities, it is safer to use the Joule as the unit of heat (1 calorie = 4.185 Joules). Nevertheless, since most textbooks and handbooks use the calorie as the MKS unit of heat, we will use the calorie in this course. (In the English system, the BTU, British Thermal Unit, is used. 1 BTU = 779 ft-lbs.)

Heat is the *quantity unit* in thermal systems (analogous to mass, charge or fluid volume in our other system types). For flow of heat we define

$$Q = \frac{dH}{dt} \quad \left(\frac{\text{cal}}{\text{sec}}\right) \tag{24}$$

3. Power - Energy

Unlike all the other systems we have studied, (across variable) times (through variable) is not equal to power flow in thermal systems. Instead, power flow is just Q. Total thermal energy transferred is

$$H = \int Q \, dt \tag{25}$$

4. Basic Thermal Elements

Thermal systems have only two basic elements' The A-type element is thermal capacitance. The D-type element is thermal resistance. There is no known T-type element in thermal systems. If it existed, the T-type element would be called thermal inductance.

4.1 Thermal Capacitance

Thermal capacitance is a measure of the capacity of an object to store heat. For an ideal thermal capacitor, we assume that a change in temperature is directly proportional to the change in the quantity of heat stored in the device. Therefore, the rate of change of temperature of an object is proportional to the rate of heat flow to the device or,

$$Q = C \frac{dT}{dt}$$

$$T(t) = \frac{1}{C} \int_0^t Q \, dt + T(0) \tag{26}$$

Note the similarity of these equations with those for electric and fluid capacitance defined previously. The units of thermal capacitance may be expressed as calories per degree C or Joules per degree C. The symbol for a thermal capacitor is the same as for the electric capacitor. Inherent in our definition above is that the temperature within the material of the capacitor is uniform. For the electric capacitor, it was reasonable to assume that all points on a metal capacitor plate were of the same potential. It is not always so reasonable to assume that all points within a volume of material are of the same temperature. However, this is true for the ideal thermal capacitance.

4.2 Thermal Resistance

A glass window pane on a winter day may have air at 20°C on one side and air at 0°C on the other. In this case, heat will be conducted from the warmer to the colder side through the glass. The rate of such heat transfer is proportional to the temperature difference across the glass (ΔT), proportional to the area of the pane (A), and inversly propotional to the thickness of the glass (L). It is also a function of the glass. In steady state, the expression for the heat flow is

$$Q = k\,(\Delta T)\,A/L \quad \text{joules/second} \tag{27}$$

k, the proportionality factor, has units joules per second-meter-degree C. This quantity is called thermal conductivity (analogous to electric conductivity). Values of k can be obtained from handbooks However, be careful of the units given in such handbook tables. Calories may be used instead of joules, hours may replace seconds, or a mixed system may be used with units such as Btu per hour-inch-degree F!

In order to carry our electric analogy further, we will define thermal resistance of a given object between two temperatures as given in Equation 28.

$$R = L/kA \tag{28}$$

Note that this is the same definition as was used for electric resistance ($R = L/\sigma A$). Then we have Ohm's Law for thermal systems:

$$T=RQ \tag{29}$$

where T is the temperature difference across a material, and Q is the flow of heat through the material. R, the thermal resistance of the material, is a function of its cross sectional area, the distance the heat must flow, and internal properties of the material given by its k value. The units of R in this case are degrees C per calorie per second.

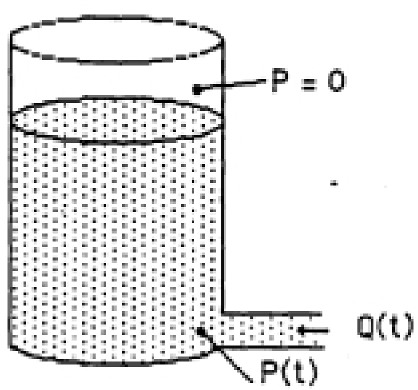
Note: A quantity commonly used to rate the heat insulating properties of materials used in buildings is the "R" factor. Unfortunately, this is not the same as the thermal resistance R we have just defined in Equations 28 and 29. The R factor of building insulating material is defined as L/k instead of L/KA as given in Equation 28. Also, mixed units are used. This R has dimensions square feet-hour-degree F per Btu. Therefore, you must divide by the total area of the insulation to get the actual thermal resistance.

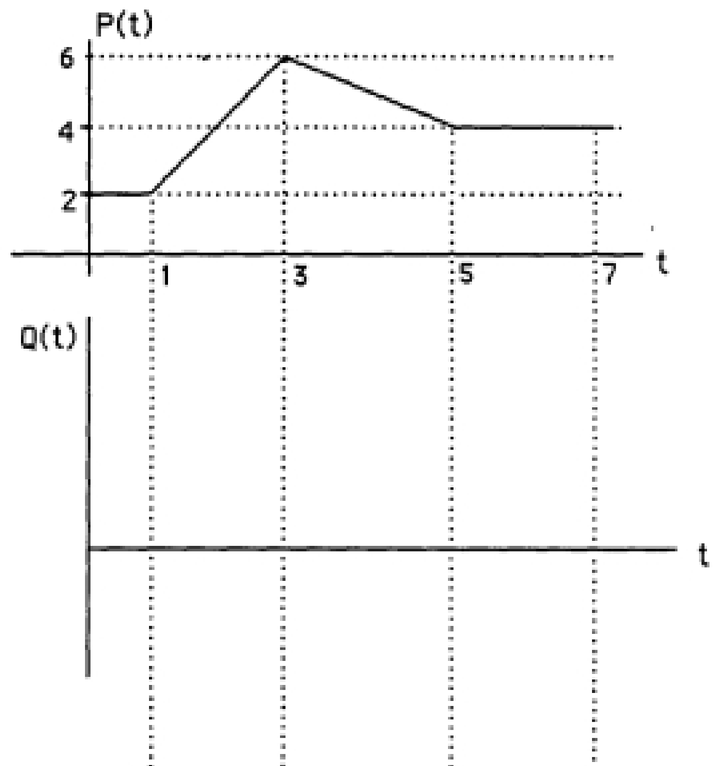
The symbol for thermal resistance is the same symbol we have used for electric and fluid resistance.

Chapter 4 Problems

1. A power transistor dissipates 15 watts when used in an amplifier output circuit. The maximum allowable internal (junction) temperature of this transistor is specified by the manufacturer to be 200 C The thermal resistance from the pn junction where the heat is developed to the transistor case is 2.5oC per watt' A "heat sink" is a metal plate on which the transistor case is mounted to aid in the flow of heat from the case to the air. Ambient temperature (temperature of the surrounding air) is 45°C. (a) Determine the maximum thermal resistance which the heat sink can possess and still meet the requirements. You tray assume the junction to case resistance and the case to ambient resistance are in series.
(b) If a heat sink of resistance 4 °C/watt is chosen, what temperature might be expected at the pn junction?

2. Consider an open reservoir being filled with water. The pressure at the bottom of the tank varies as shown in the figure (pressures are in n/m2). The tank has a fluid capacitance of 2 ms/n.
Sketch the graph of the flow rate, Q (m3/s), into the tank as a function of time. Label all important values on your sketch! Write an analytic function for this flow in terms of step functions. What is the power flow at t = 4? Is it into or out of the system?





Systems Chapter 5 Study Guide

Generalizations and Analogies

A. **Concepts Addressed By This Topic**

I' Generalized variables:
across, through, integrated across, integrated through.

2. Generalized ideal one-port elements
Energy storage, energy dissipative, energy sources.

3. Generalized through and across sources
4. Alternative effort/flow definitions

B. Introduction

During our study of mechanical, electrical, fluid and thermal systems to this point, certain analogous elements, equations, and concepts should have become obvious. We now wish to tie these together into one generalized approach which applies to any of these systems and to other systems as well. We will define the generalized ideal system elements according to their energy handling characteristics. The variables will fit into two categories: across and through.

System elements can also be categorized according to the number of energy ports they possess (port is another name for a terminal pair). The ideal elements we have introduced are all one-port elements. Power flow at each port is the product of that port's across and through variables. Ideal one-energy-port elements can be classified as active (sources of unlimited energy), or passive (elements which either store or dissipate the energy entering at their port). Those passive elements which store energy can be further divided into two classes according to the variable (across, A or through, T) which indicates the energy provided or stored. Ideal sources may also be classified as across (A-type) or through (T-type) based on which variable they maintain at their terminals. When an ideal source has its A or T variable set as a function of some other through or across variable of the system, it is called a *dependent* source. Dependent sources are useful in modeling more complex system components such as amplifilers, transformers, transducers, etc.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Identify the through variable, the integrated through variable, the across variable, and the integrated across variable used with mechanical, electrical, and fluid systems.
2. Sketch the symbol used for the ideal T-type and A-type energy storage elements, and the D-type energy dissipative element, along with the ideal elemental equation for each, for each system type studied to date (see Table 4-2).
3. Represent forcing functions by the appropriate source symbol. Model real sources using a combination of ideal elements.

D. Study Procedure

Read Chapter 5.
Additional material on the standardization of nomenclature and symbols can be found in references 5 and 11.

E4 Systems
Chapter 5
Generalizations and Analogies

Certain common concepts and definitions have appeared in all the systems we have studied. In this chapter we summarize these ideas and use them to define a more general set of system elements and relations.

1. Across and through variables.

Two primary system variables have been associated with each system element: an across variable and a through variable. We will adopt the designation v for the generalized across variable, and f for the generalized through variable. v21 represents the difference between the values of the across variable at terminals 2 and 1. That is, v21 = v2 - v1

Table 1 Across and Through Variables by System Type

System	Across variable	Through variable
General Case	v	f
Translational Mechanical	v: velocity	F: force
Rotational Mechanical	ω: angular velocity	T: torque
Electrical	v : voltage	i : current
Fluid	P: pressure	q: volumetric flow rate
Thermal	T: temperature	Q: heat flow rate

2. Integrated across and through variables.

Pure system elements (mass, inductance, resistance, etc.) may be defined by the algebraic relations between forms of across and through variables. These equations are called constitutive relationships. In some cases, the relationship uses the integral of the across or through variable.

Integrated Across Variable	Integrated Through Variable
$x = \int v \, dt$	$h = \int f \, dt$

$$x = \int v \, dt \qquad h = \int f \, dt \tag{1}$$

Table 2 Integrated Across and Through Variables

System	Integrated across variable	Integrated through variable
General Case	**x**	**h**
Translational Mechanical	**x**: displacement	**p**: momentum
Rotational Mechanical	θ: angular displacement	**h**: angular momentum
Electrical	λ: flux linkage	**q**: charge
Fluid	Γ: pressure momentum	**V**: volume
Thermal	none	**H**: heat

3. One-port elements.

All system elements defined so far have been one-ports. This means the element has one pair of terminals by which energy can be transferred between it and the rest of the system. The across variable may be designated as v21, with terminals l and 2 specified, or without a subscript if the terminals are marked + and -. In this latter case, it is understood that v represents v+ - v-. The direction of the through variable is indicated by an arrow.

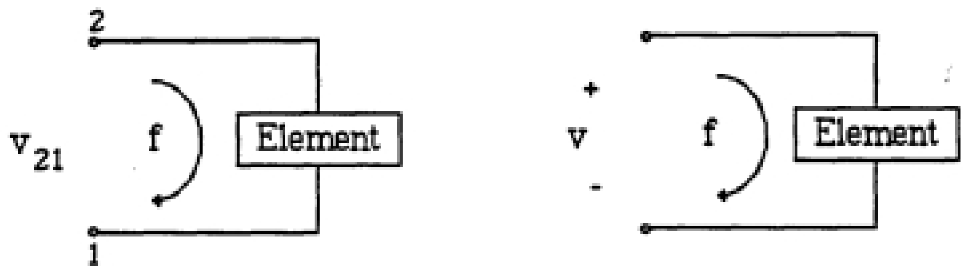


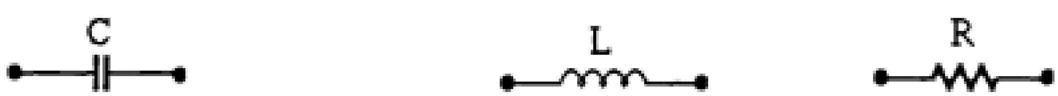
Figure 1 A- and T- variable conventions for a basic element

The equations we use to relate the across and through variables for each element assume the conventions of Figure 1. That is, when we write v21 = f R, we are assuming that the positive direction chosen for f is through the element **from 2 to 1.**

4. A-type, T-type, D-type elements.

Three element types have been defined based on their energy handling properties. In Ttype elements, energy stored is a function of the through variable. In A-type elements, energy stored is a function of the across variable. In D-type elements, energy transferred to the element is dissipated as heat. We may use the following symbols for these elements, regardless of the system type:

A-Type **T-Type** **D-Type**



$$f = C\frac{dv_{21}}{dt} \qquad v_{21} = L\frac{df}{dt} \qquad v = f\,R$$

$$v_{21} = \frac{1}{C}\int_0^t f\,dt + v_{21}(0) \qquad f = \frac{1}{L}\int_0^t v_{21}\,dt + f(0) \qquad f = \frac{1}{R}v_{21} \tag{2}$$

The symbols used earlier for mechanical system elements (given in Figure 2) may still be used for clarity on occasion. However, the generalized symbols for A-, D-, and T-type elements of Figure 1 will now be the primary ones used to represent mechanical elements as well as elements **from all** system types.

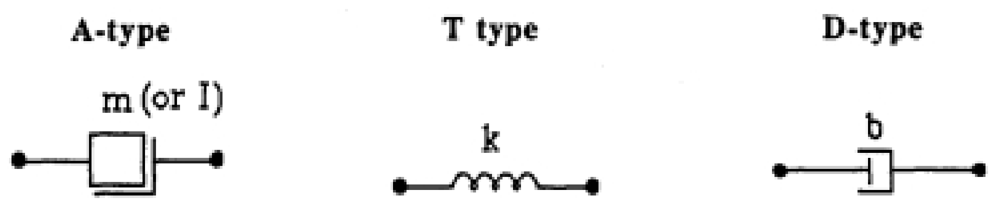


Figure 2 Alternative Mechanical System Element Symbols

Two important distinctions exist between the element equations for the mechanical parameters of Figure 2 and those for the generalized parameters given by Equations 2: L in the generalized T-type element equation is numerically equal to 1/k, the reciprocal of the mechanical spring constant, and R of the generalized D-type element equation is numerically equal to 1/b, the reciprocal of the mechanical damping constant.

Table 3 Relations Among A-, T-, and D-type parameters

System	A-type parameter	T-type parameter	D-type parameter
General Case	C	L	R
Translational Mechanical	m: mass	1/k: spring	1/b: damper
Rotational Mechanical	I: rotational mass	1/k: rot. spring	1/b: damper
Electrical	C: capacitor	L: inductor	R: resistor
Fluid	C: fluid capacitor	I: . fluid inductor	R: fluid resistor
Thermal	C: thermal capacitor	none	R: thermal resistor

5. Constitutive Relationships

The constitutive relations for C and L are obtained by replacing (f dt) by dh and (v_{21} dt) by dx in the integral form of Equations 1 and 2, yielding algebraic equations in terms of the integrated through and across variables, respectively.

Integrated Through Variable	Integrated Across Variable
$\int f\,dt = \int C\,dv$	$\int v\,dt = \int L\,df$

Constitutive relations: $h = C\,v$ $\qquad$ $x = L\,f$

The constitutive relation for R is $\qquad v = R\,f \qquad$ (3)

6. Power and energy

For all systems studied (except thermal), power associated with a given element is the product of the element's across and through variables:

$$p(t) = (\text{across})\ (\text{through})$$

Note that power is an instantaneous quantity, meaning that the rate of energy flow at any instant depends only on the values of the across and through variables at that instant. Previous history has no significance. Power can be positive or negative' Following the standard conventions (Figure 1) for across and through variables, positive power indicates energy flow into the element (to be stored or dissipated), and negative power indicates energy flow out from the element (to the rest of the system).

The power flow to a D-type element is

$$p = v f = f^2 R = \frac{v^2}{R} \quad \text{watts} \tag{4}$$

Energy transferred to or from an element over a period of time may be found by using the time integral of power. The total energy transfer over all time to R, L, or C elements must be positive or zero, since none of these elements is able to generate its own energy. Energy transferred to A-type (C) or T-type (L) elements is stored and may later be returned to the system Energy transferred to a D-type element (R) is always lost to the system. The energy stored in an A-type element is determined by the square of its A variable; the energy stored in a T-type element is determined by the square of its T variable.

A-Type Element T-type Element

$$E = \frac{1}{2} C v^2 \qquad\qquad E = \frac{1}{2} L f^2 \quad \text{joules} \tag{5}$$

7. Sources

A source element is a one-port element which is capable of supplying unlimited energy to a system. An ideal source is one which maintains a prescribed output under any loading conditions. An *independent* source is one in which the prescribed output is not a function of any other system variable' Sources may either deliver or absorb power. The power associated with a source is p = v21 f. When the flow is directed out of the terminal of higher v, the source is delivering energy to other elements in the system.

7.1 A-type source

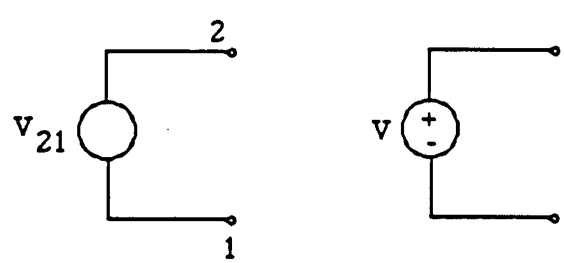


Figure 3 A-Type Source Symbols

An A-type source maintains a prescribed value (v21 of Figure 3) of the across variable between its terminals, regardless of other elements connected there. The designation of terminals 2,1 may also be accomplished using +/- signs' A summary of A-type sources of systems we have studied are given in Table 4:

Table 4 A-Type Sources

General Case	v: general across source
Translational Mechanical	v: Translational velocity source
Rotational Mechanical	ω: angular velocity source
Electrical	v : voltage source
Fluid	P: pressure source
Thermal	T: temperature source

7.2 T-type source

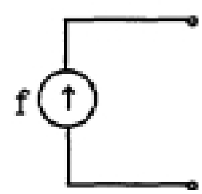


Figure 4 T-Type Source Symbol

A T-type source maintains a prescribed value (f in Figure 4) of its through variable, regardless of other elements connected at the terminals. For the systems we have studied, T-type sources are

Table 5 T-Type Sources

General Case	f : general through source
Translational Mechanical	F: force source
Rotational Mechanical	T: torque source
Electrical	i: current source
Fluid	q: *flow* source
Thermal	Q: heat *flow* source

7.3 **Dependent sources**

The sources of sections 7'1 and 7.2 are *independent* sources' That is, their output is completely described by the v or f function supplied with the source. It is also possible for an ideal A- or T-type source to have its output controlled by other variables in the system. For example, consider a voltage amplifier which receives a voltage signal from a microphone and delivers an amplified version of that signal to a loudspeaker. If the amplifier does its job, the voltage at its output is K times the input voltage. If we want to represent the output of the amplifier as a voltage source driving a speaker, we must set the value of the voltage source equal to K times the input voltage from the microphone. In other words, the voltage source is *dependent* on the microphone signal. We might represent this amplifier as shown in Figure 5.

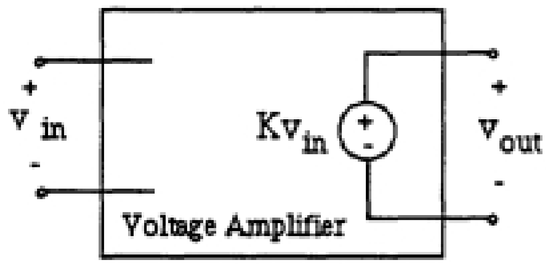
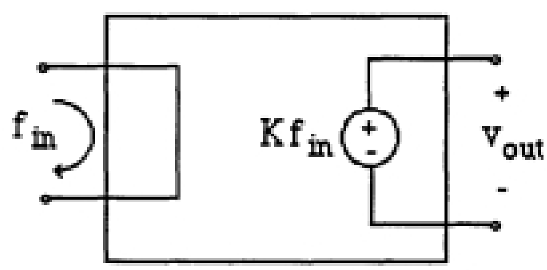


Figure 5 Voltage Amplifier

In this case, the input control (microphone) was considered an across variable, vin. The controlling variable could also be a through variable. For example,



Current controlled voltage source

Figure 6 Current Controlled Voltage Source
(Note that the K values must have proper

More convenient symbols for *dependent* sources are shown in Figure 7. The T-type dependent source symbol is on the left, and the A-type dependent source symbol is on the right. v or f represents a variable somewhere else in the system. Note that both the T-type and the A-type dependent sources may be functions of an across or a through variable elsewhere in the system. This type of source is very convenient when modeling such real devices as control valves, transistors, amplifiers etc., where a small signal controls the flow from a larger power source.

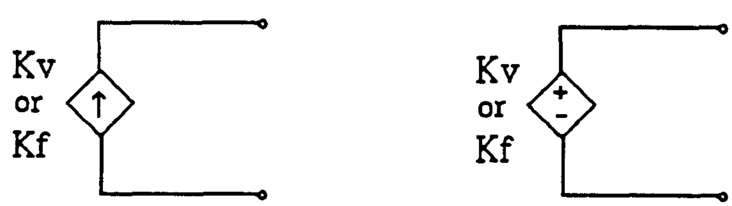


Figure 7 Dependent Source Symbols

8. Alternative categories for system variables

The prime aim of our analysis of systems is to keep track of energy transfers. To this end, we have found in each of the mechanical, electrical, and fluid system types, a pair of variables whose product is power flow. So that analogies may be drawn across all these system types, we have categorized one variable of each system's pair as an across variable, and the other as a through variable. These names were chosen simply because the physical nature of each variable makes its type easy to determine. It does not indicate, for example, that velocity and voltage are somehow alike. By categorizing the variables this way, we can translate a set of equations reIating the A- and T-type variables of one type system into equations representing another type system with the same solution. This categorization of variables is sometimes called the *Firestone analogy* after the person who first called attention to it.

A second, equally correct and useful way to categorize variables is the mobility' analogy. In this approach, we think of the variables force, torque, voltage, and pressure as "effort" variables. That is, they represent quantities which have a potential to produce motion. The variables linear velocity, angular velocity, current, and fluid flow are called "flow" variables because they each represent a movement of mass or charge. Using these definitions, the effort variable in electric or fluid systems is what we have defined as the across variable. The flow variable is what we have defined as the through variable. For the mechanical systems, however, the correspondence is reversed: the effort variable is the through variable and the flow variable is the across variable'

Using concepts of effort and flow (instead of across and through) to make analogies between system types does introduce a new set of rules, diagrams and mathematical relations to be used. For one example inductance, rather than capacitance becomes the electrical analog of mass. Table 6 lists the assignments of variables under the mobility analogy.

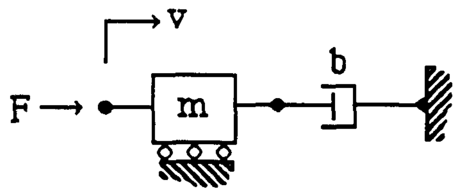
Table 6 Effort and Flow Variables

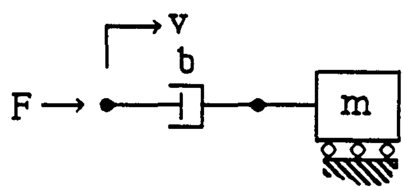
System	Effort variable	Flow variable
Translational Mechanical	F: force	v: velocity
Rotational Mechanical	T: torque	w: angular velocity
Electrical	v: voltage	i: current
Fluid	P: pressure	q: volumetric flow rate
Thermal	T: temperature	Q: heat flow rate

Although the system diagram may be different, the same set of equations will result when numerical values of the parameters are substituted, and the same answers to any systems problem will be found. The effort/flow concept is not uniformly better or worse than the across/through idea. Both have certain applications where they are superior. We will not study the effort/flow approach further in this course. However, you should be familiar with these terms and be able to identify the variables they refer to for each system.

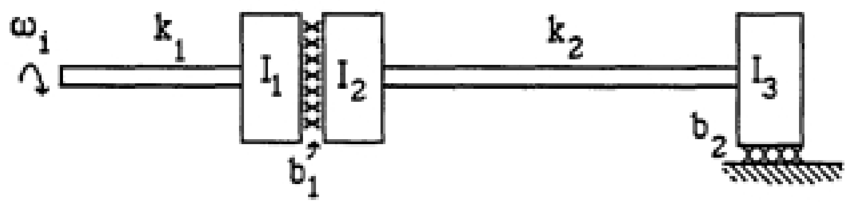
Chapter 5 Problems

1. For each system below, $m = 2kg$, $b = 5$ N-s/m.
   (a) Sketch a graph of the system using generalized ideal symbols. Label element values.
   (b) Write an equation which relates F and v.





2. Sketch a graph of the rotational system below using generalized elements. ω_i is the input function. Label each element value using proper form of K, I, and b.



Systems Chanter 6 Study Guide

Singularity Functions and Initial Conditions

A. Concepts Addressed By This Chapter

1. Singularity functions as signals.
2' Derivatives and integrals of singularity functions'
3. Signals as a combination of singularity functions.
4. Response of ideal elements to singularity driving functions.
5. Initial (t = 0+) conditions after t = 0 application of singularity functions'

B. Introduction

An input signal we must deal with in every system type we study is the simple two level or on-off signal. This signal can represent a suddenly applied force or torque, a switch closing in an electric circuit, a sudden valve opening or closing, etc. A mathematical function has been defined to represent this signal, called the unit step function. Each system type we have studied possesses elements which respond to an input signal by producing the derivative or integral of the input. Therefore, we need to investigate the derivatives and integrals of the unit step function. Since the step function has a discontinuity, its derivatives are mathematically undefined at that point. However, it is convenient to use a limiting process to define a family of functions called singularity functions which can be used to handle the discontinuity problem. This chapter will introduce these functions and illustrate their application to system elements.

C. Instructional Objectives

A student who masters this material will be able to

1. Write the definition and graph of step, ramp, and impulse functions. Express a given appropriate waveform analytically in terms of these functions.
2. Given an expression composed of singularity functions in time, sketch a graph of the value of the expression vs. time
3. Determine the response of ideal elements to step, ramp, or impulse inputs.
4. Use the sifting property of the impulse function to evaluate integrals and other expressions containing this function.
5. Replace elements which possess initial energy storage with equivalent combinations of . elements with no energy storage together with an equivalent source.
6. Determine the initial (t = 0+) state of all system energy storage elements after an application of a singularity function source at t = 0.

D. Study Procedure

Read Chapter 6.
Additional information on singularity functions can be found in references 1, 6, 9, and 13.

E4 Systems
Supplementary Notes
Singularity Functions and Initial Conditions

1. **Unit Step Function**

Certain time function signals appear often in the study of systems. One of these, the unit step, may be thought of as a model of the signal which appears when a switch is closed connecting a constant source to output terminals.

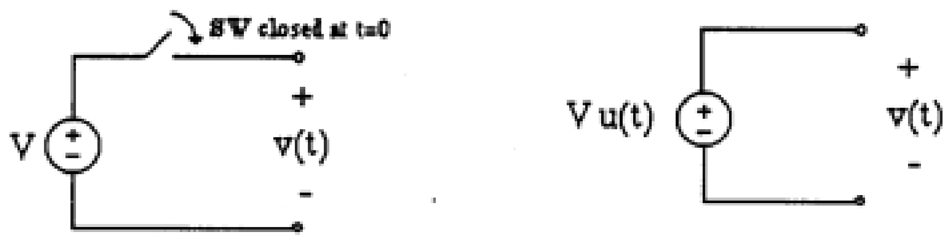
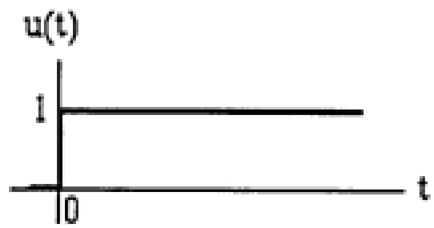


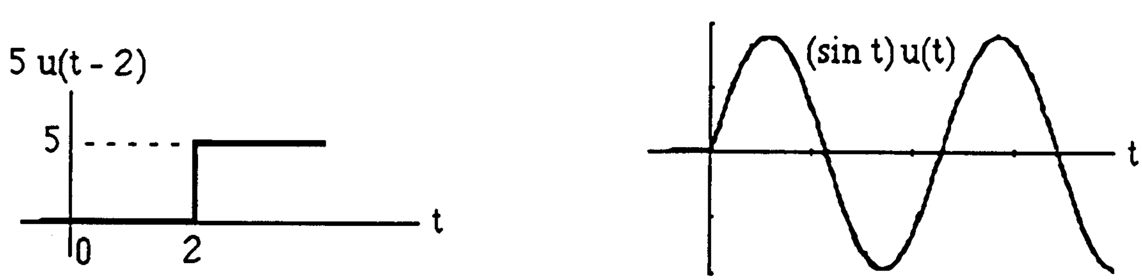
Figure 1 Replacement of a switch by a unit step function source

The definition of the unit step function, u(t), is given by Equation 1.

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \qquad (1)$$



The step function can be translated on the time axis and can be multiplied by any value or other function as illustrated by the examples of Figure 2:



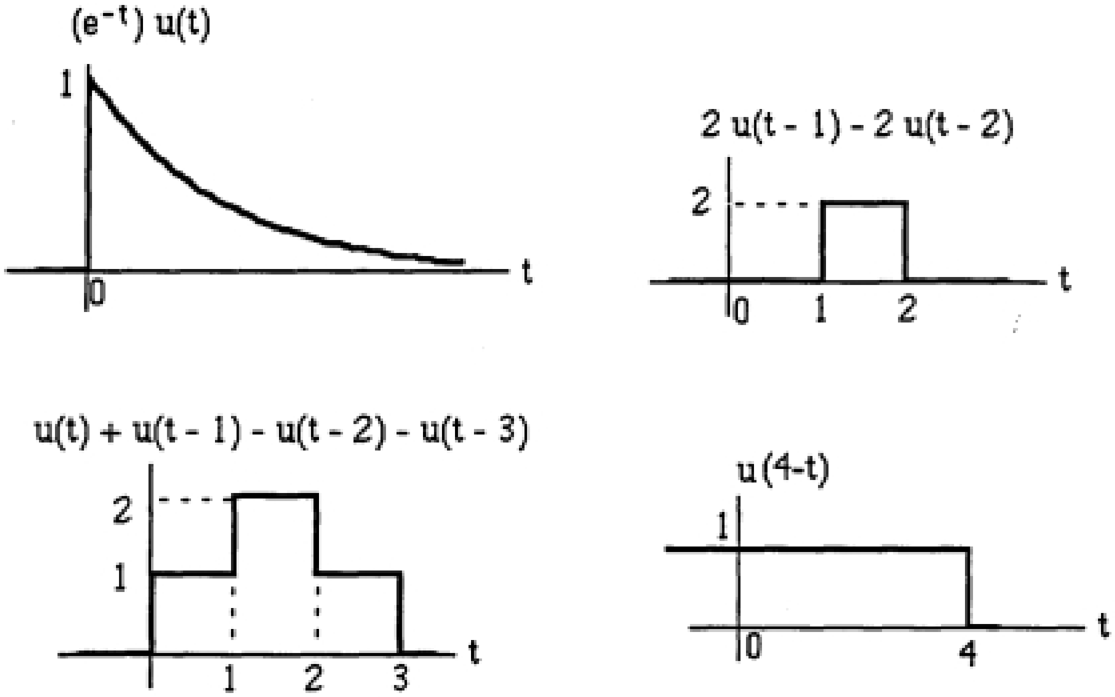


Figure 2 Combinations of the unit step with other functions

2. Unit Ramp Function

We will use r(t) to indicate the unit ramp function. (In other textbooks, you may find other symbols used for the unit ramp.) Its definition is

$$r(t) = \begin{cases} 0, & t<0 \\ t, & t\geq 0 \end{cases} \tag{2}$$

So a plot of the the unit ramp consists of two straight line segments, one on the negative time axis, and one of unit slope for positive t.

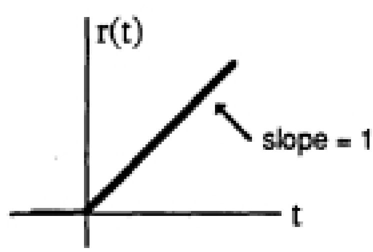
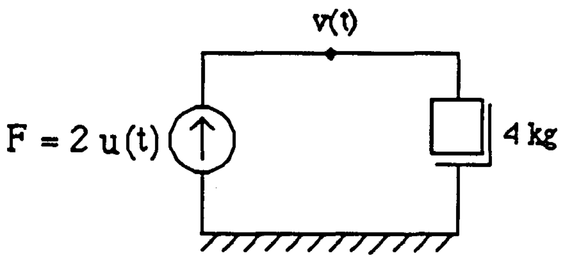


Figure 3 Unit Ramp Function

The unit ramp is related to the unit step. If we integrate u(t) from $-\infty$ to t, the result (the area accumulated under the unit step function) is r(t). We can define r(t) in terms of u(t):

$$r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda \qquad \text{or} \qquad r(t) = t\, u(t) \tag{3}$$

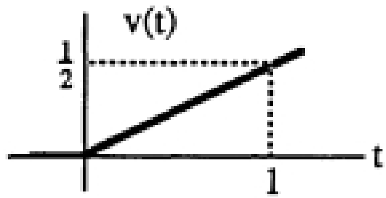
Example: A 2N force is applied to a pure mass of 4 kg at rest at t=O. Find its velocity for all t.



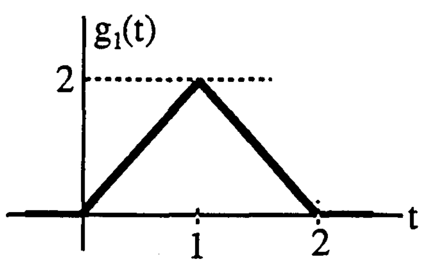
$$v = \frac{1}{C}\int_{-\infty}^{t} f(t)\, dt + v(0)$$

$$= \frac{1}{4}\int_{-\infty}^{t} 2\, u(t)\, dt = \frac{1}{2}\, r(t)$$

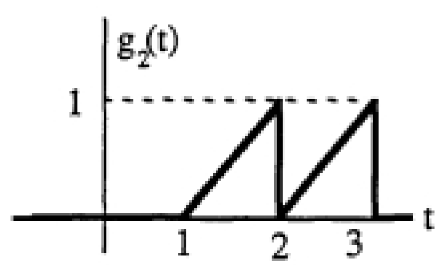
Result:



Ramp functions may be combined to produce other shapes:



$g_1(t) = 2\, r(t) - 4\, r(t-1) + 2\, r(t-2)$



$g_2(t) = r(t-1) - u(t-2) - u(t-3) - r(t-3)$

Figure 4 Combinations of ramp functions
Note that the coefficient of each ramp function represents the change in slope.

Excel : Assume g 1(t) of Figure 4 is a time-varying torque applied to a pure rotational spring element. Determine the angular velocity across the spring as a function of time if the spring constant, k = ½.

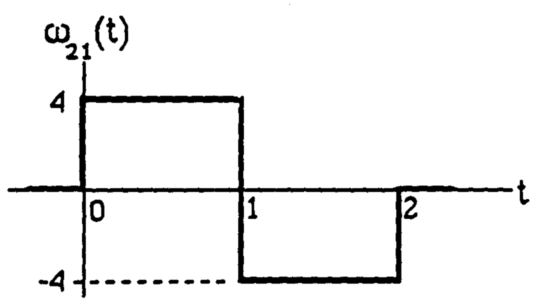
Solution: Use the spring element relation between torque and angular velocity:

$$\omega_{21} = \frac{1}{k}\frac{dT}{dt} = 2\frac{d}{dt}\left[g_1(t)\right]$$

Differentiate the waveform $g_1(t)$:

$$\omega_{21} = 2\frac{d}{dt}\left[2\, r(t) - 4\, r(t-1) + 2\, r(t-2)\right]$$

$$= 4\, u(t) - 8\, u(t-1) + 4\, u(t-2)$$



3. **Unit Impulse Function**

Suppose a step current is applied to a pure inductor (i = (t)). Then

$$v = L\frac{di}{dt} = L\frac{d}{dt}\left[u(t)\right]$$

But what is the derivative of u(t)? To approximate this function, consider an approximation of a unit step function current as given in Figure 5:

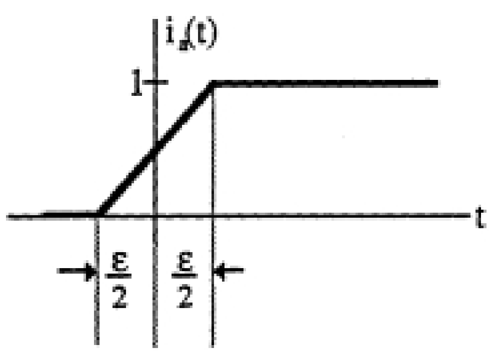


Figure 5 Approximation of a step.

As a approaches zero, the function of Figure 5 becomes more and more like a step. An expression for ia(t) of Figure 5 can be written using ramp functions. Its derivative then is a function of steps.

$$i_a(t) = \frac{1}{\varepsilon}\ r(t - \frac{\varepsilon}{2}) - \frac{1}{\varepsilon}\ r(t + \frac{\varepsilon}{2})$$

$$\frac{di_a}{dt} = \frac{1}{\varepsilon}\ u\,(t - \frac{\varepsilon}{2}) - \frac{1}{\varepsilon}\ u(t + \frac{\varepsilon}{2}) \tag{4}$$

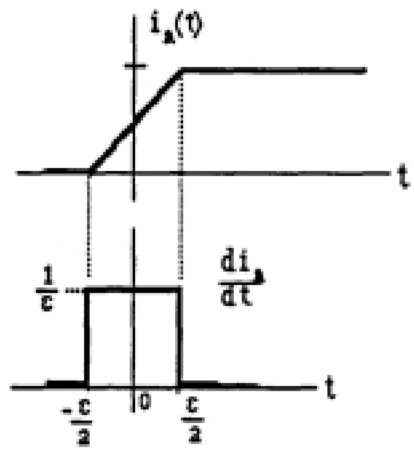


Figure 6 Approximation of a unit step (ia(t)), and its derivative.

di_a/dt is a rectangle of height $1/\varepsilon$, and width ε. Now let ε get very small. As $\varepsilon \to 0$, $i_a(t)$ looks more and more like a unit step function. Meanwhile, derivative becomes a very narrow, very high rectangle, but with area always equal to 1.

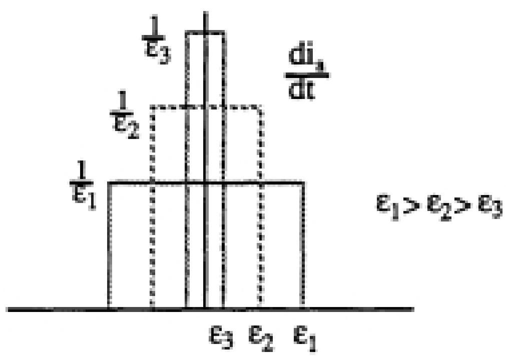


Figure 7 Derivatives of $i_a(t)$ of Figure 6 as ε is made smaller.

A special symbol, $\delta(t)$, is used for the function which results for $\varepsilon \to 0$.

$$\delta(t) = du(t)/dt. \tag{5}$$

One way to define δ(t) (which may not be mathematically satisfying)

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \qquad\qquad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \tag{6}$$

Because of the symbol, the function is sometimes called the delta function (also the Dirac delta function). It is impossible to sketch a graph of δ (t) since it has infinite value at t = 0. Therefore, we draw an arrow pointing to ∞ at the point on the axis where the non-zero value appears. Beside the arrow, in parentheses, we indicate the area of the impulse' On the left in Figure 8 we see the diagram for a unit impulse function located at the origin. From Equation 6, this is designated δ (t). The right panel of Figure 8 illustrates the effects of multiplication of the impulse by a constant (the area is multiplied), and time shifting.

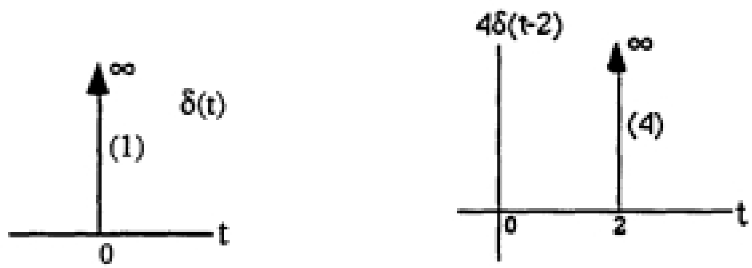


Figure 8 Plots of δ(t) and 4 δ(t - 2)

δ (t) is zero everywhere but at the origin' Any integral between definite limits which include the origin yields the value 1 because this is the area under the impulse function. Integrals whose definite limits do not include the point where the impulse occurs will have the value zero. Of course the impulse may be located at any point on the time axis.

If the integral is indefinite (limits -∞ to t for example), the integral of the unit impulse function is the unit step.

$$\int_{-\infty}^{t} \delta(t)\, dt = u(t)$$

As used in systems calculations, the unit impulse will always appear as part of an integrand' Real systems always possess energy storage elements and are therefore non-instantaneous (they have memory). Input-output equations for such systems always involve terms which integrate the input. When the duration of an input function is much shorter than the response time of a system, that input function may be replaced inside the integral by an impulse of equal area. You have encountered this approach before in mechanics.

Example: The mass m, at rest on a frictionless plane is hit by a hammer. The hammer and mass are in contact for a very short time during which the mass is suddenly accelerated to velocity v. After hammer and mass part, the velocity of the mass will continue to be a constant, v.

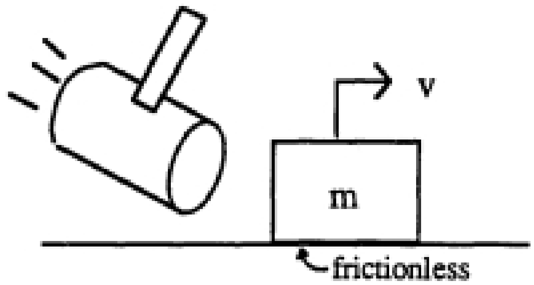
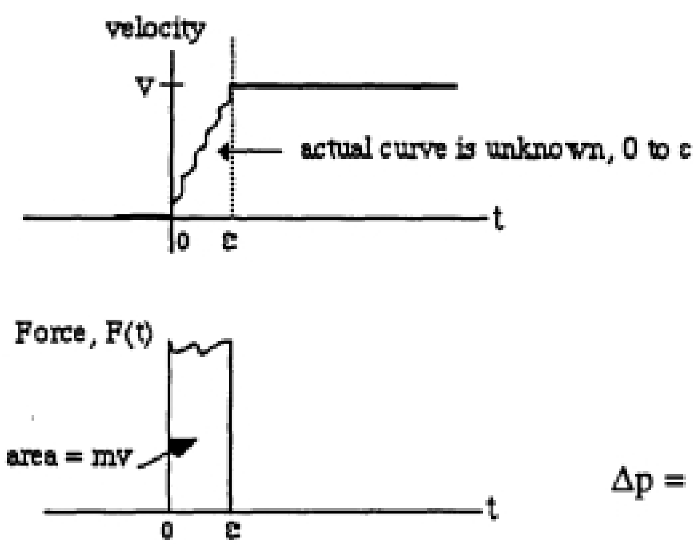


Figure 9 Hammer hits initially motionless mass.

For such a sudden change in the state of the system, we are not usually concerned with the actual force or velocity functions during the time of contact, only with the total changes that took place. Assume mass and hammer are in contact from $t = 0$ to $t = \varepsilon$. We know that for $t < 0$, the velocity was zero, and for $t > \varepsilon$, the velocity is a constant, v. We can therefore draw a graph of velocity vs. time for the mass for $t < 0$ and $t > \varepsilon$. During $0 > t > e$, the velocity follows some path from 0 to v. The exact shape of this path is usually not critical to the solution of problems of this type. Therefore, on the first graph of Figure 10, we sketch a guess of this part of the picture. Since only mass is present, the force the hammer exerts on the block is m dv/dt. This is plotted in the second graph.



$$\Delta p = \Delta mv = \int_0^{\varepsilon} F\, dt$$

Figure 10 Velocity of mass (top) and force exerted by hammer (bottom). .

Since v was initially zero, the area under the F(t) curve must total my (the total change in momentum is the area under the graph of the force function). If a is very small compared to the system's memory (this system remembers the hit forever), we can approximate the force F(t) by an impulse function:

$$F(t) = mv\ \delta(t) \tag{g}$$

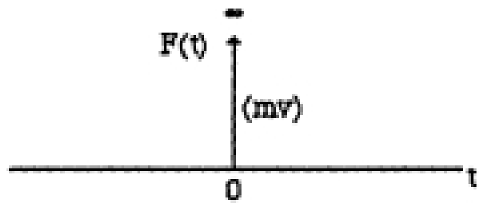


Figure 10 Approximation of the force applied in Figure 9

3.1. **Special properties of the impulse function**

Since the impulse function almost always appears as part of an integrand, we need to know the procedure for evaluating functions such as

$$\int_{-\infty}^{+\infty} \delta(t)\, x(t)\, dt$$

or, more generally,

$$\int_{-\infty}^{+\infty} \delta(t - a)\, x(t)\, dt \tag{9}$$

where x(t) is a continuous function of t at t = a. The two functions are plotted in Figure 11.

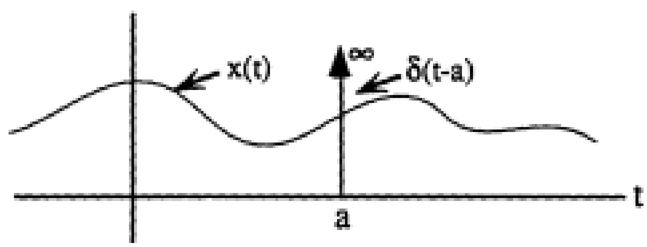


Figure 11 Plots of the integrand of (9)

The two functions are multiplied together, so their product is zero for all t except t = a. Therefore, we can shrink the limits of integration from $\pm\infty$ to the interval $a-\varepsilon$ to $a+\varepsilon$.

$$\int_{a-\varepsilon}^{a+\varepsilon} \delta(t - a)\, x(t)\, dt$$

If ε is very small, the change in x(t) over the range a-ε to a+ ε is negligible and we can take x(t) = x(a), a constant for the range of integration.

$$\int_{a-\varepsilon}^{a+\varepsilon} x(a)\ \delta(t - a)\ dt \ = \ x(a) \int_{a-\varepsilon}^{a+\varepsilon} \delta(t - a)\ dt \ = \ x(a)$$

The last step simply evaluates the area under the impulse as 1. The property of the impulse function under an integral sign to select one value of a function it multiplies is called the sifting or sampling property: The integral of the product of an impulse function and x(t) is x(t) evaluated at the point where the impulse function occurs.

The examples below illustrate the behavior of the impulse function under various circumstances.

$$\int_{-\infty}^{+\infty} e^{-2t}\ \delta(t)\ dt \ = \ 1 \qquad\qquad \int_{0}^{4} r(t)\ \ \delta(t - 2)\ dt \ = \ 2$$

$$\int_{0}^{\pi} \sin x\ \ \delta(x - \frac{\pi}{4})\ \ dx \ = \ \frac{1}{\sqrt{2}} \qquad\qquad \int_{0}^{1} r(t)\ \ \delta(t - 2)\ \ dt \ = \ 0$$

$$\delta(t)\ \cos t = \delta(t) \qquad\qquad e^{at}\ \delta(t - 1) = e^{a}\ \delta(t - 1)$$

$$\delta(t)\ \sin t = 0 \qquad\qquad (\sin \pi t)\ \delta(t - \frac{1}{2}) \ = \ \delta(t - \frac{1}{2})$$

$$\delta(t)\ \delta(t - 1) = 0$$

4. System State and Initial Conditions

The "state" of a system is a set of data about the system at a particular time, to, which makes it possible to determine the value of all variables of the system at t = to and (given all input functions) the value of all variables of the system for t > to. For systems comprised of only D-type elements and known sources, all A and T variables can be found by writing and solving a set of algebraic equations. Therefore, to specify the state of such a zero order system at t = to, no information beyond the configuration of the system is needed. However, if energy storage elements are present in the system, additional information will be required.

Consider a mass-spring system with a zero forcing function. Such a system has a sinusoidal natural response. However, the amplitude and phase position of the sinusoid depend on how we start the motion at t = 0. If we stretch the spring and hold the mass still, then let go, the velocity may be close to V_{max} sin wt That is, the mass has zero velocity at the instant we let go, then it begins to move upward. The amplitude V_{max} is determined by the initial energy we store in the spring (a T-type element, so a function of the force). If we start with an unstressed spring, and start the motion with a push on the mass, the velocity may be closer to V_{max} cos wt Here, the mass starts off at maximum velocity, determined by the energy stored by the push (an A-type element, so a function of velocity). Therefore to solve for the velocity of this system under any starting conditions, we need information about the initial energy storage of both the spring and the mass. Two items of information are required to completely specify a second order system.

In general, one way of specifying the state of a system at t = to is to list the energy stored in each A- or T- type element. The equivalent of this is to specify the A variable of all A-type elements and the T variable of all T-type elements. If we want to determine system behavior over some time period, say, $t = 0$ to ∞, a determination of the state of the system at $t = 0$ (the initial conditions) is required.

In many problems, the initial values of the variables in the energy storage elements is specified. However, a common situation involves the opening or closing of a switch which changes the system configuration at $t = 0$. Assuming we know the system state just before the switching, how can we find it just after the switch action (the new initial conditions)? The energy stored in an A- or T-type element is a function of its A or T variable, respectively. Could this energy ever be discontinuous? That is, could a plot of energy stored in an A- or T-type element ever have a step-type change from one value to another as shown below?

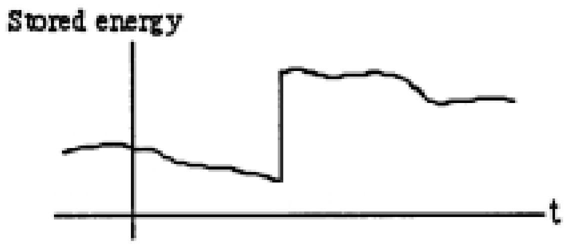


Figure 12 Discontinuity of stored energy

Since power flow to any element is the derivative of energy stored, a step change in stored energy represents an infinite (impulse) power flow. Power is always the product of the A and T variables of the element. Therefore, to have an impulse power flow to an element, either its A or T variable must be an impulse. We can assume that infinite energy is not possible in real systems. This means that the A variable cannot be infinite in an A-type element and the T variable cannot be infinite in a T-type element. (There are some circumstances under which impulse A variables may appear across T-type elements and/or impulse T variables may appear through A-type elements.)

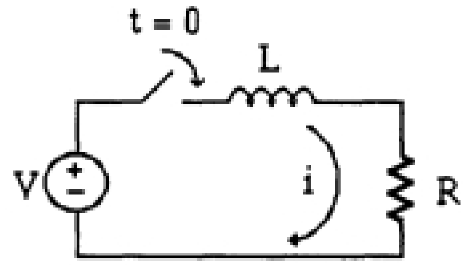
Based on the foregoing discussion, we can state the following rule which is very useful in determining conditions in systems just after a switch is thrown:

If power flow is finite (in other words, in the absence of any impulses) the A variable in each A-type element and the T variable in each T-type element cannot change instantaneously.

Table 1 Variables which are continuous in the absence of impulses

T-Type	**A-Type**
Force of spring	Velocity of mass
Torque of rotational spring	Angular velocity of rotating mass
Current of inductor	Voltage of capacitor
Flow of fluid inductor	Pressure of fluid capacitorExample:

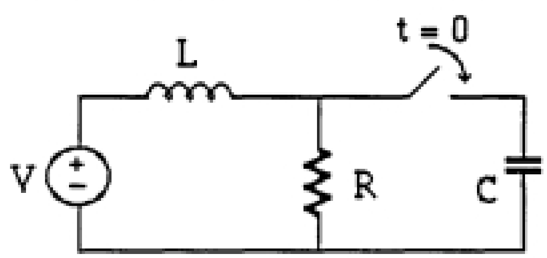
Close the switch at $t = 0$ on the R-L system shown. Find $i(0^+)$ and di/dt at $t = 0^+$. (0^+ indicates the instant of time just after the switch closes. 0^- indicates the instant of time just before the switch closes.)



Solution:

Before the switch was closed, current in the system, and therefore current in L was zero. That is, $i(0^-) = 0$. According to the rule above, just after the switch is closed the inductor current must still be zero (no impulse voltages are present). So $i(0^+)$ is also zero. Now, since the resistor has no current at $t = 0^+$, there is zero voltage across the resistor. Therefore the battery voltage must be entirely across L at $t = 0^+$. This means that at $t = 0^+$, $V = v_L = L\, di/dt$. So the initial value of di/dt is V/L.

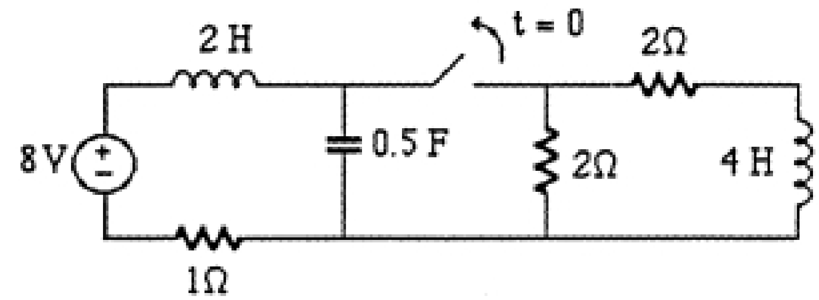
Example: The system shown below was in operation for a long time with the capacitor uncharged when, at $t = 0$, the switch was closed. Find the current in C at $t = 0^+$.



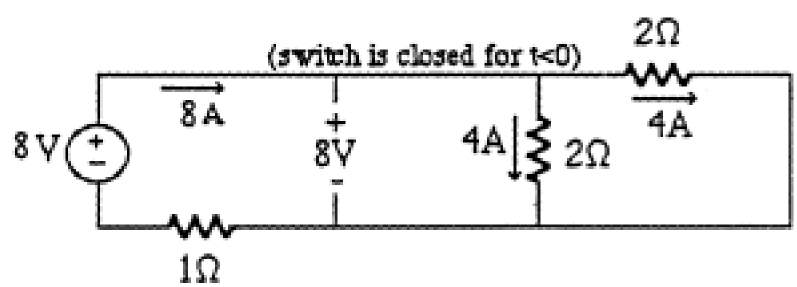
Solution: We observe that there should be no impulse currents or voltages in this system at $t = 0$. Only two variable values existing at $t = 0^-$ are of any use to us in determining conditions at $t = 0^+$. These, according to Table 3, are the inductor current and the capacitor voltage. $v_C(0^-)$ is given as zero so $v_C(0^+)$ must also be zero. To find $i_L(0^-)$, we can write an equation around the only loop existing during $t < 0$: $V = L\, di/dt + Ri$. In equilibrium, the derivative is zero, so $i_L(0^-) = V/R$. This means that $i_L(0^+) = V/R$. Now, at $t = 0^+$, since the capacitor voltage is zero, the resistor voltage must also be zero (they are in parallel for $t>0$). If $v_R = 0$, then i_R must also be zero. If the resistor has zero current, all the inductor current must be flowing through C at $t = 0+$. Therefore $i_C(0^+) = i_L(0^+) = V/R$.

When you are told that a system is in equilibrium at $t = 0^-$, it means the system has been in operation for a long enough time that all exponentially decaying (natural form) A and T variables have died out. Then, if only constant forcing functions are present, you can assume all variables have reached constant values and the time derivative terms in any differential equation you might write are zero. Since vL = L di/dt, and iC = C dv/dt, we can assume this means that all inductor voltages are zero (elements with current flow but zero voltage could be drawn as a short circuit), and all capacitor currents are zero (elements with voltage but zero current could be drawn as an open circuit).

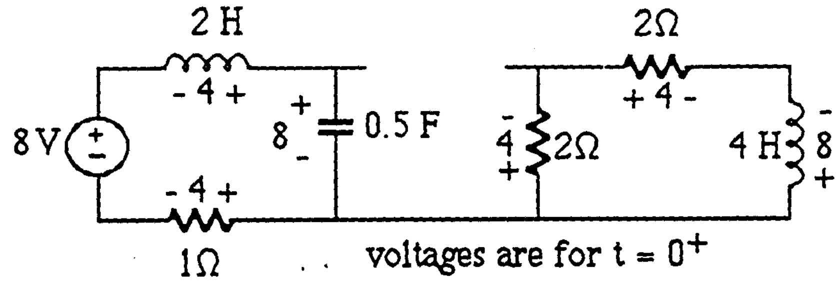
<u>Example:</u> The circuit below is in equilibrium when, at $t = 0$, the switch is <u>opened</u>. Find all currents and voltages at $t = 0^-$ and at $t = 0^+$.



<u>Solution:</u> For the determination of equilibrium values at $t = 0^-$, replace all capacitors by open circuits and inductors by short circuits:



The currents and voltages of the resulting resistor system are easily found, and this solutic also gives us the C and L currents and voltages. Now open the switch.



At t = 0+, the current around the left loop must be 4 because the inductor current does not change. This means all three elements in the left loop have i = 4 at $t = 0^+$. Since the 1Ω resistor current is 4, its voltage is also 4. The capacitor voltage must remain 8 at $t = 0^+$. The path law then says $v_L(0^+) = 4$ with polarity as shown. For the right-hand loop (now disconnected from the left) the inductor current must remain 2. So the loop current is 2, giving each resistor a voltage of 4. This means the inductor voltage must be 8 at $t = 0^+$. (Remember, these values pertain only to one instant, $t = 0^+$).

5. Modeling of initial conditions

To solve the differential equations of any system for t > 0, the state of the system at t = 0+ must be known. By state is meant a set of t = 0+ values of across and/or through variables from which all other variables of the system may be found. One such set is those variables which determine the energy stored in each storage element: the across variable of each A-type element, and the through variable of each T-type element. These values appear as constants (f(O-) or v(0-) in the element equations good for t>0:

$$f(t) = \frac{1}{L}\int_{0^-}^{t} v_{ab}(t)\,dt + f(0^-) \qquad \text{T - type} \tag{10}$$

$$v_{ab}(t) = \frac{1}{C}\int_{0^-}^{t} f(t)\,dt + v(0^-) \qquad \text{A - type} \tag{11}$$

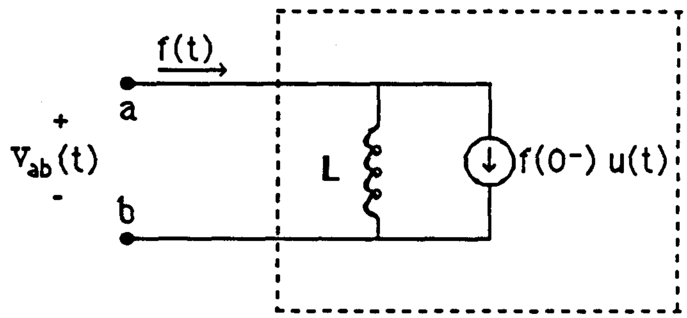
The reason for the use of 0- in the above equations is to take care of the possibility that the integrand may contain an impulse at t = 0. In that case, we need to start the integral just before the impulse so that the impulse area will be recorded.

It is worth taking a moment here to look more closely at the effect an impulse function has on an energy storage element. Since infinite energy is assumed not possible in any real system, we rule out impulse functions for the A variable in an A-type energy storage element, and for the T-variable in a T-type energy storage element. Based on Equation 10, if we apply an impulse Vab(t) of value $K\,\delta(t)$, f(t) jumps at t = 0 from $f(0^-)$ to $[f(0^-) + K/L]$. Similarly, from Equation 11, if we apply an impulse f(t) of area K at t = 0, the value of v(t) will jump at t = 0 from $f(0^-)$ to $[f(0^-) + K/L]$. We can update our rules concerning impulses applied to energy storage elements in light of the above results:

1. An impulse T variable applied to an A-type energy storage element results in a step change of the A variable equal to 1/C times the area of the impulse.
2. An impulse A variable applied to a T-type energy storage element results in a step change of the T variable equal to 1/L times the area of the impulse.
3. Because it implies infinite energy, impulse functions are not allowed for the A variable across an A-type element, or for the T variable through a T-type element.

4. In the absence of impulses, the A variable across an A-type element, and the T variable through a T-type element must be continuous functions of time (no step changes).

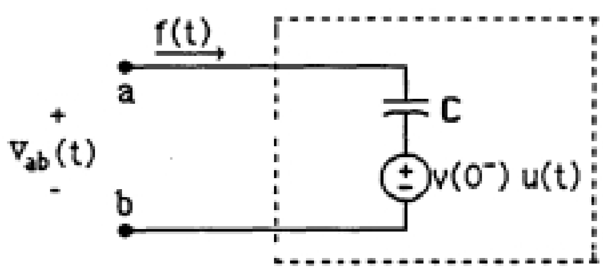
We have introduced models for each pure element, but these models do not include information about initial conditions. Such information can be included in the model by the addition of a through or across type source as shown below. In Figure 13, an equivalent model of a T-type element is shown. The dotted outline represents the boundary of the T-type element of Equation 10. This model will produce the same result as Equation 10 for t > 0. The pure L element is assumed to have zero flow at t = 0-. If no impulse v is applied, it will still have zero flow at t = 0+. However, the source provides the correct total flow as specified by Equation 10, f(0-), at the terminals. If an impulse is applied at t = 0, the flow at t = 0+ will be the source value f(0-), plus
the step change through L caused by the impulse, that is, [f(0-) + K/L] where K is the impulse area.



$$f(t) = \frac{1}{L}\int_{0^-}^{t} v_{ab}(t)\,dt + f(0^-)\,u(t)$$

Figure 13 Equivalent circuit for a T-type element

Now consider Figure 14 representing an A-type element. The value of the across variable, v(t), from a to b is the sum of the v difference across C and the v difference across the source. If the flow is f(t), the first term on the right of Equation 11 gives the v difference across C. The second term is equal to the source for t > 0. Therefore, Figure 14 is an equivalent model of an across element with non-zero initial conditions for t > 0.

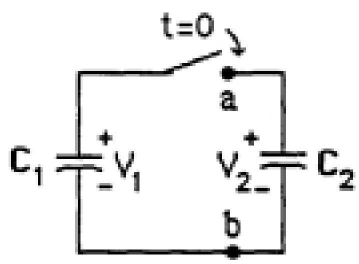


$$v_{ab}(t) = \frac{1}{C}\int_{0^-}^{t} f(t)\,dt \;+\; v(0^-)u(t)$$

Figure 14 Equivalent circuit for an A-type element

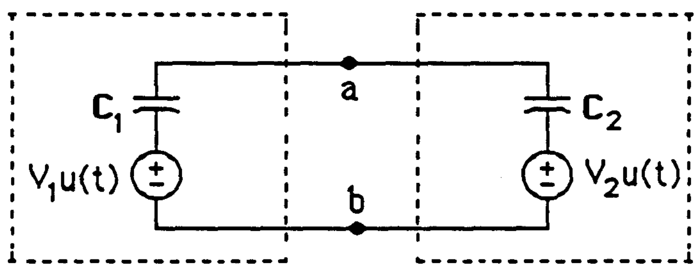
Example:

Two capacitors, C1 and C2, charged to different voltages, V1 and V2, are connected together at t = 0. What is the final voltage across each?

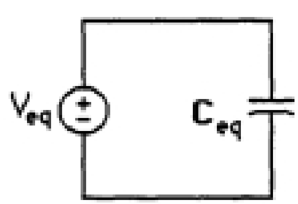


Solution:

Replace each C by its initial condition equivalent:



Now the Cs are in series, as are the sources. Combining series elements gives: Veq= (V l -V2) u(t) acting on Ceq = C1C2/(C1+C2).



The total charge transferred is then

q = CeqVeq= (C1C2 / IC1+C2]) (V1 -V2).

This charge flows through both capacitors. To find vab, we see that it is the voltage across

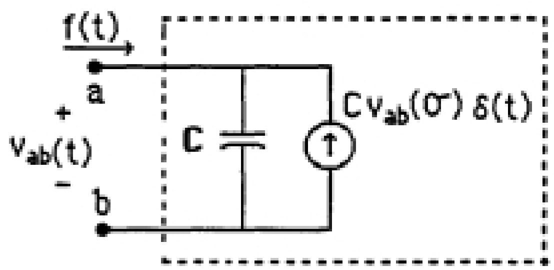
C1 plus the source V1, or across C2 plus source V2. Let's use C2 (all voltages are given for for t>0):

$$vC2 = q/C2 = C1\ (V1 - V2) / (C1 + C2)$$
$$vab = VC2 + V2 = C1V1 + C2V2) / (C1 + C2)$$

A second form of initial condition model can be derived for each storage element type. The equivalent circuit for the across element and its equation are given in Figure 14. Differentiate through this equation and solve for f(t):

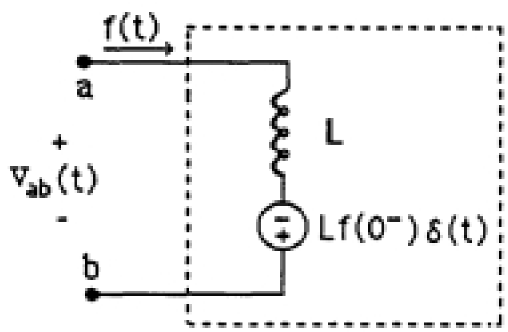
$$f(t) = C\ dv_{ab}/dt - C\ v_{ab}(0^-)\ \delta(t) \tag{12}$$

Equation 12 can be modeled as a pure C whose flow variable is given by the first term on the right, in parallel with an upward directed flow source whose value is the second term.



Similarly, a second equivalent for the T-type element can be obtained by differentiating the equation of Figure 10 and solving for v(t):

$$v_{ab}(t) = L\ df/dt - L\ f(0^-)\ \delta(t) \tag{13}$$



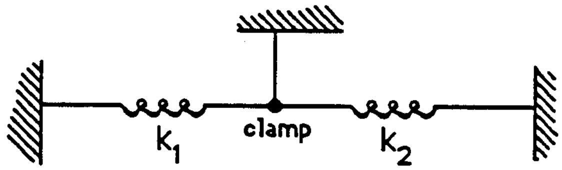
<u>Direction of the initial Condition Source</u>

For the electric and fluid storage elements we have studied, knowledge of the initial conditions includes the direction of voltage or pressure gradient across A-type elements and current or flow direction through T-type elements. Therefore, the direction for the initial condition source in the equivalent is clear. However, in other systems this direction may not be so obvious. For the translational mechanical spring element for example, the initial condition is compression receives an inward force from elements at either end. What should then be the direction of the step function flow source of Figure 10?

A look at Figure 10 shows that the flow source is in the direction of the external f(t) when the spring is being compressed (vab positive). Therefore, the rule for such flow source equivalent models is: Assume a positive direction for force (or torque) flow through the system. Springs in the system should have their initial condition force flow in this assumed flow direction when in compression, and opposed to the assumed flow when in tension.

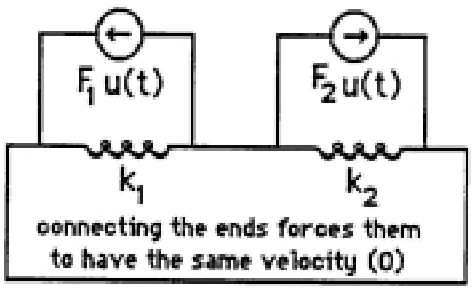
Example:

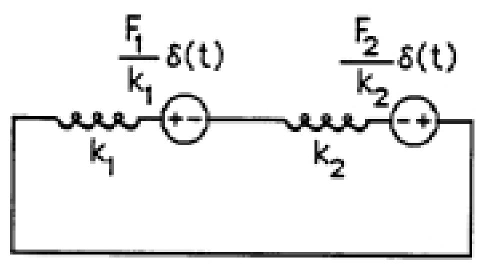
The springs are initially held by a clamp which causes spring 1 to be compressed by force F I and spring 2 to be stretched by force F2. The clamp is then released. Find the final force in the spring. (Note that the force will be the same in both springs the instant the clamp is removed.)



Solution:

Assume a positive direction for force flow left to right. Then the equivalent flow source models are as shown on the left below, since spring 1 is compressed and spring 2 is stretched. In this case, the solution is easier if we switch to the velocity source models on the right.





The springs are in series, giving an equivalent spring constant of $L_{eq} = k_1k_2/(k_1+k_2)$. The sources are also in series, so they add, giving an equivalent velocity source of $F_1\delta(t)/k_1 - F_2\delta(t)/k_2$. The resulting force in the combined spring when this velocity difference is imposed is:

$$f(t) = 1/L_{eq} \int v(t)\, dt \;=\; k_1k_2/(k_1+k_2) \int (F_1/k_1 - F_2/k_2)\ \delta(t)\, dt$$

Since the integral of $\delta(t)$ is $u(t)$, the (compressive) spring force, $f(t)$, for $t > 0$ is the constant:

$$f = (k_2F_1 - k_1F_2) / (k_1 + k_2)$$

Chapter 6 Homework

1. Evaluate the following integrals:

(a) $\int t^2\,\delta(t-2)\,dt$ (b) $\int \cos \pi x\,\delta(x-1)\,dx$

(c) $\int [\delta(t) - \delta(t-1)]\,dt$ (d) $\int [\delta(t)\,\delta(t-1)]\,dt$

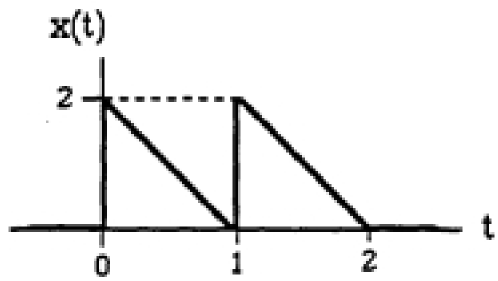
2. (a) Sketch f(t) and its first derivative, where

$f(t)=r(t)-u(t-1) - u(t-2)-r(t-2)$.

(b) Sketch the integral of $\delta(t-1) + \delta(t-2) - 2\,\delta(t-3)$.

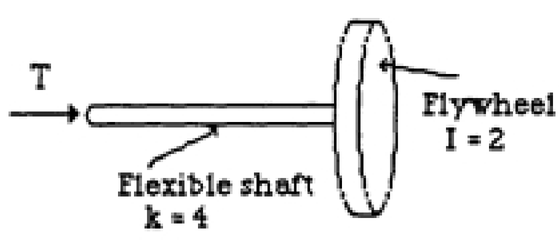
3. Determine the current which would flow in a pure capacitor of C=2F if a voltage of $v = r(t) - u(t-1) - u(t-2) - r(t-2)$ is applied to its terminals.

4. (a) Write an analytical expression for the function x(t) plotted below.

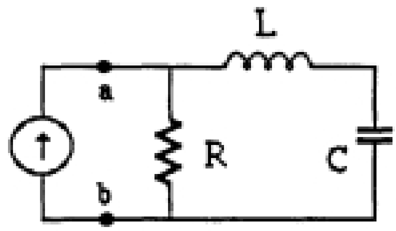


(b) Write an expression for and sketch the derivative of x(t) of part (a).

5. The system shown was at rest when, at t = 0, a constant torque T = 12 N-m was suddenly applied to the left end of the flexible shaft. Determine the angular velocities of both ends of the shaft at t = 0+. Also determine the total angular twist of the shaft at t = 0+.



6. The system shown was "dead" at t = 0-. The current source puts out a 1 amp step, u(t). Determine the initial (t=0+) values of the following quantities: (a) Capacitor voltage (b) Resistor voltage (c) di/dt through the inductor.



,Systems Chapter 7 Study Guide

Analysis of Elementary Dynamic Systems

A. Concepts Addressed By This Topic

1. **Modeling** of real systems using ideal elements.
2. Rules of Compatibility and Continuity.
3. Path and Vertex conditions.
4. Loop and Nodal methods of analysis.
5. Source Transformations
6. System Theorems

B. Introduction

Modeling: We are now ready to obtain sets of equations which describe (approximately) the behavior of real dynamic systems. The procedure is first to model the system using the ideal elements we have defined, then to obtain a set of independent system equations from the model. In many cases, reducing the system to a model is -the most difficult part of the process. It involves representing each of the real system's components (e.g., wheels, belts, motors, etc. of a mechanical system) with one or more ideal elements (mass, spring, damper, source).

After the proper ideal elements are chosen and their interconnection has been determined, certain conditions may be applied to obtain equations which relate the variables of the system. The conditions include compatibility, continuity, the path rule, and the vertex rule. Loop and nodal analysis methods are two of the possible methods of application of these conditions to obtain sufficient equations to solve the system.

There are several procedures of model simplification which may be used prior to the application of nodal or loop techniques. One simplification process involves replacing elements including an ideal source of one type (A or T) with equivalent elements containing an ideal source of the other type. Replacement of larger sections of a system by only two equivalent elements is possible using certain theorems (Thevenin's Theorem, Norton's Theorem). Another theorem, the maximum power transfer theorem, is useful in determining the proper load to connect to the output of a system to assure that maximum will be delivered by the system to the load.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Given a real mechanical, electrical, fluid, or thermal system, construct an equivalent model using interconnected combinations of ideal elements.

2..Given a system model containing ideal elements including sources, determine the number of independent equations which can be obtained using (a) the path condition and (b) using the vertex condition.

3. Using the path or vertex conditions, obtain an independent set of equations sufficient to solve for either the across or the through variables of the system (loop or nodal methods).

4. Use source transformation to replace an ideal A source, element combination with an equivalent combination using an ideal T source and vice-versa.

5. Replace any one-port system with its Thevenin por Norton equivalent.

6. Determine the load which will draw maximum power from a system.

D. Study Procedure

Read Chapter 7 of these notes.
Additional information can be found in References 1, 9, and 11.

Chapter 7

Analysis of Elementary Systems

We are now ready to obtain sets of equations which describe (approximately) the behavior of real dynamic systems. The procedure is first to model the system using the ideal elements we have defined, then to obtain a set of independent system equations from the model. In some cases, reducing the system to a model is the most difficult part of the process. It involves representing each of the real system's components (e.g., wheels, belts, motors, etc. of a mechanical system) with one or more ideal elements (mass, spring, damper, source).

After the proper ideal elements are chosen and their interconnection has been determined, the equations relating the variables of the stystem may be determined by applying certain rules governing the connection of ideal elements. These rules will now be developed.

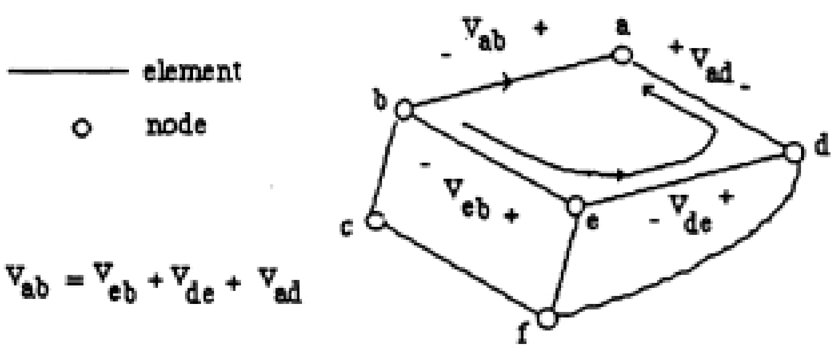
1. Rules for Connected Elements

1.1 Compatibility-. (across type variables) When terminals of two ideal elements are physically connected, a constraint is placed on the across variable at that terminal. For example, in a mechanical system, terminals connected together must stay together in motion, and therefore both must have the same velocity with respect to any other point in the system. Analogously, in an electrical system, connected terminals must have the same voltage; in a fluid system, the same pressure; in a thermal system, the same temperature. This is called the condition of compatibility.

System	Quantity connected element terminals have in common
Mechanical	Linear or angular velocity
Electrical	Voltage
Fluid	Pressure
Thermal	Temperature

Note that all the quantities above are A-type variables.

1.2 Path Condition: Since the across variables at connected terminals must be equal, the difference in the value of the across variable between any two system nodes, i and j, may be found by adding algebraically the difference across each element in any path between nodes i and j. Since the across difference between nodes i and j is one value, this sum must be the same regardless of the path chosen. If we travel from i to j by one path and return via another path, the sum of variable differences across the elements of this closed path is equal to zero. This is known as the *path condition* (for electric systems, it is called Kirchhoff's voltage law).



$$V_{ab} = V_{eb} + V_{de} + V_{ad}$$

Figure 1 A connected system of 8 elements and 6 nodes.

Figure 1 shows 8 elements connected between 6 nodes. (Each element is indicated here simply by a line.) The across variable for an element connected between nodes i and j is expressed as vij. If we start at node b and travel through the element that connects from b to a, we will travel through a total increase of va - vb = vab . (Of course, if vb exceeds va, vab will be negative.) Suppose we travel from b to a by a different route, b-e-d-a. The total observed rise in the across variable on this trip is veb + vde + vad. This sum must also be equal to vab since we again have traveled from b to a. Our conclusion is that the sum of the v changes along any path from node i to node j is the same. From this we can formulate a rule known as the path *condition:*

<u>The algebraic sum of the A-Me variables around any closed path must equal zero.</u>
(This rule in electric systems is often called Kirchhoffs Voltage Law.)

How many independent path equations can be written for a system? Consider a system as in the figure below (dots are nodes, lines between dots are system elements). Panel a shows only the N system nodes. In b, we add elements (represented by lines called "branches") in such a way that all nodes are brought into a common "tree" connection with no closed paths formed. This always requires N-1 branches. After the tree is constructed, each additional branch added completes a new closed path as shown in panel c. Therefore, if there are B branches in the final system, the number of independent paths are B - (N-1) or: The number of Independent Path Equations = B - N + 1.

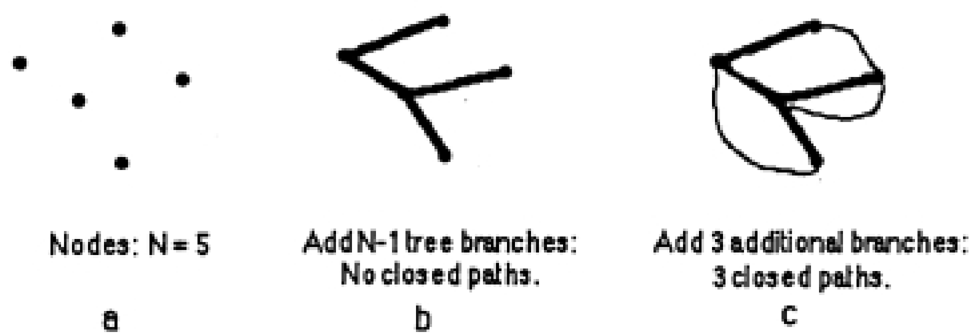
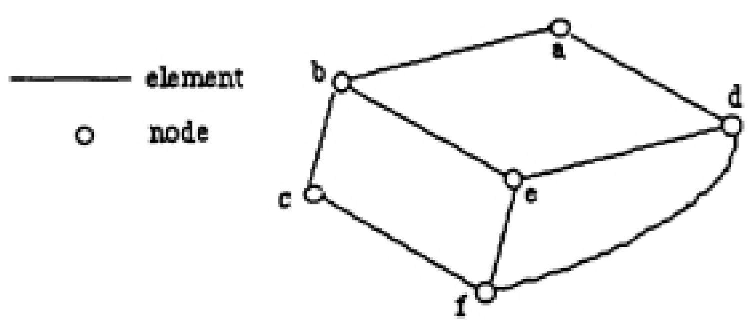


Figure 2 The number of independent closed paths is B - N + 1

Example: Given a system for which there are 8 elements and 6 nodes as shown below. Write as many path (loop) equations as possible without passing a node twice. How many of these equations are independent?



Answer. Seven path equations can be written without *involving a* node twice:

vbc+vcf+vfe+veb=0
vde+vef+vfd=0
vbc+vcf+vfd+vde+veb=0
vab+vbe+vef+vfd+vda=0
vab + vbc + vcf + v fd + vda= 0
vab+vbe+ved+vda=0
vab+vbc+vcf+vfe+ved+vda=0

According to the B - N + 1 rule, only 8 - 6 + 1= 3 of these are independent (e.g. subtraction of the last two gives the first). To find an independent set, always choose B-N+1 paths so that all branches are included and so that each path includes a branch not covered by the previously chosen paths.

1.3 Continuity: (through type variables) The through variable of each system type we have studied is the time derivative of another variable *known as* the integrated through variable

Through Variable	Integrated Through Variable
force, F	linear momentum, p
torque, T	angular momentum, h
current, i	charge, q
flow rate, Q	volume of fluid, V
heat flow, Q	heat energy, H

$$F = \frac{dp}{dt} \qquad T = \frac{dh}{dt} \qquad i = \frac{dq}{dt} \qquad Q = \frac{dV}{dt} \qquad Q = \frac{dH}{dt}$$

The integrated through variables are all quantities which, by laws of physics, are conserved in any interaction between system elements. In other words, the rate at which momentum, charge, etc., leaves one element must at all times equal the rate at which it enters connected elements so that the total remains constant. Remember, this rate of change is the through variable. The *condition is* known as the condition of *continuity.*

1.4 Vertex Condition: Consider a node of a system to which four elements are connected as shown in Figure 3.

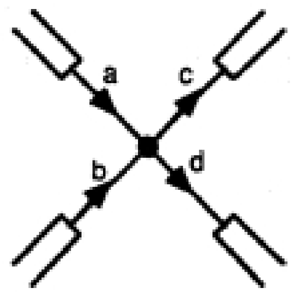


Figure 3 Vertex Condition: t + fb - fc - fd = 0

If momentum or charge or volume or heat is flowing in the four elements in the directions shown, we can sort the elements into two groups by the direction of their flow (to or from the node). We can then use continuity to state that the sum of the through variables of group 1 (entering the node) must equal the sum of the through variables of group 2 (leaving the node).

Vertex Condition: The sum of all through variables entering a node is zero. (In electric systems, this is known as Kirchhoff's current law.)

How many independent vertex equations can we write? Consider the system of Figure 4.

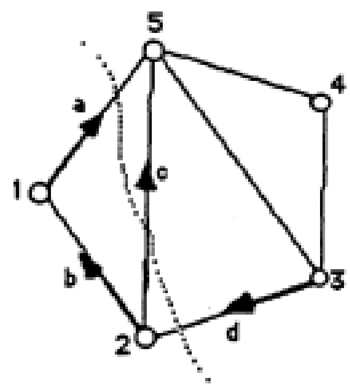


Figure 4 The sum of flows across any line which cuts through the system is zero.

If we write vertex equations at every node, each branch flow appears in two equations, once with a + sign and once with a - sign. If we add the equations of two nodes, the flow in any element connected between these nodes cancels out. For example, at node 1, flows -a + b = 0. At node 2. flows -b + -c + d = 0. Adding these equations gives -a - c + d = 0. The three flows of this last expression cross the dotted line in the figure, indicating that the vertex condition also applies to "cuts" such as this line, through the system. If we add the equations for all nodes but one, say node 5, all flows appearing twice will cancel, leaving the equation for node 5. In general, adding the equations for N-1 nodes gives the equation for the Nth node. Therefore, there are N-1 independent vertex law equations.

Number of Equations Required to solve the System

The unknowns of any system are the across and through variables of each element. Assuming B elements in the system, we have 2B unknowns. Combining independent Path and Vertex equations, we have (B - N + 1) + (N -1) = B equations. But we also know a relation between the across and through variables of each element. These provide B additional equations. Therefore the path and vertex conditions are sufficient to solve the system.

2. Analysis Methods

2.1 Nodal Analysis

(For illustrative purposes, we will develop this technique using flow sources and resistive elements. Later we will show that the procedure applies foe all sources and element types.) Consider the system of Figure 5 in which there are 4 nodes. We know it is possible to write 4 - 1 = 3 independent vertex equations for this system. The most efficient way to do this is first to choose one node (any one - usually the bottom) as the reference, indicated by the symbol ⏚.

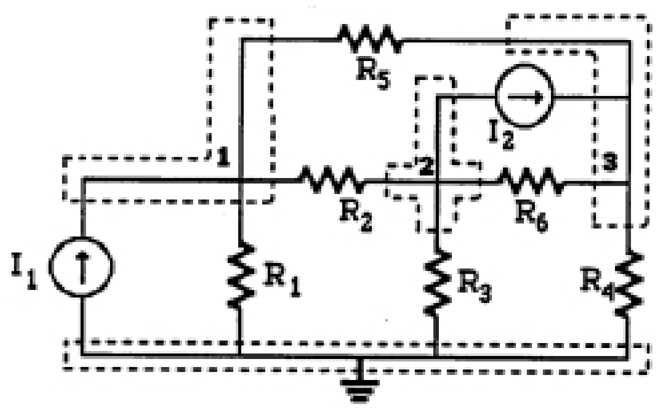


Figure 5 System with 4 nodes requires 3 node equations

To simplify the equations, we use conductance, G =1/R. The equations will set source currents into the node (left side) equal to resistor currents leaving the node (right side). For example, at node 1 there is a current I1 entering the node from the source, and currents leaving the node via G1, G2, and G5.

$I l = G 1 v 1 + G2 (v 1- v2) + GS (v l - v3)$ Node 1

$-I2 = G2 (v2 - v 1) + G3 (v2) + G6 (v2 - v3)$ Node 2

$12 = GS (v3 - vl) + G6 (v3 - v2) + G4 v3$ Node 3

(1)

The equations are easier to solve if we collect the v I, v2, and v3 terms:

$I1 = v 1 (G l + G2 + GS) - v2 (G2) - v3 (GS)$

$-I2 = - v 1 G2 +v2 (G2+ G3 + G6) - v3 (G6)$

$12 = - VI(G5) - v2(G6) +v3 (G4 + GS + G6)$

(2)

This is more efficiently written in matrix form:

$$\begin{bmatrix} G_1+G_2+G_5 & -G_2 & -G_5 \\ -G_1 & G_2+G_3+G_6 & -G_6 \\ -G_5 & -G_6 & G_4+G_5+G_6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ -I_2 \\ I_3 \end{bmatrix} \qquad (3)$$

The conductance matrix on the left of Equation 3 can be obtained directly from the system diagram without performing the steps of Equations 1 and 2. Name the elements of the matrix as shown below:

$$\begin{bmatrix} G_{11} & -G_{12} & -G_{13} \\ -G_{21} & G_{22} & -G_{23} \\ -G_{31} & -G_{32} & G_{33} \end{bmatrix}$$

Based on Equation 3, we can establish rules for the direct determination of the elements of this matrix:

Gii is the sum of all conductances touching node i,
Gij is the sum of all conductances common to nodes i and j.

Note that the main diagonal terms are all positive, and all off-diagonal terms are negative. This will always be the case for nodal method equations.

This method may also be used to obtain simultaneous differential equations for systems where energy storage elements are present. Consider the system of Figure 6.

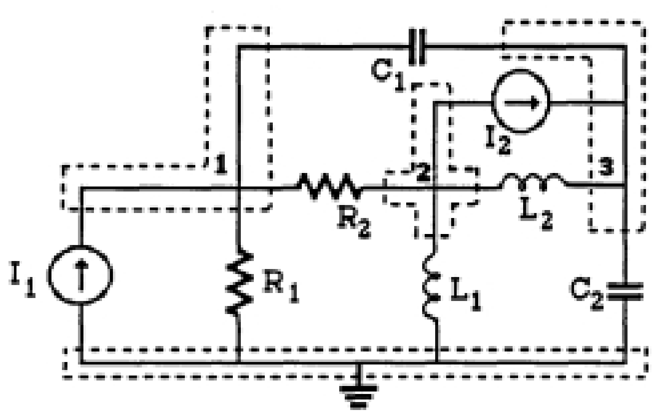


Figure 6 Node method applied to a system containing R, L, and C

(Remember that $i_C = C\dfrac{dv}{dt}$ and $i_L = \dfrac{1}{L}\int v\,dt$)

Define an operator, D, which produces derivative with respect to time, and D' 1 which . produces integral over time. Use these symbols to write the nodal equations for the above network:

$$I_1 = G_1(v_1) + G_2(v_1 - v_2) + C_1 D(v_1 - v_3)$$

$$-I_2 = G_2(v_2 - v_1) + (1/L_1)D^{-1}(v_2) + (1/L_2)D^{-1}(v_2 - v_3)$$

$$I_2 = C_1 D(v_3 - v_1) + (1/L_2)D^{-1}(v_3 - v_2) + C_2 D(v_3)$$

Writing these equations in matrix form produces the following matrix of the system elements:

$$\begin{bmatrix} G_1+G_2+C_1D & -G_2 & -C_1D \\ -G_2 & G_2+\frac{1}{L_1}D^{-1}+\frac{1}{L_2}D^{-1} & -\frac{1}{L_2}D^{-1} \\ -C_1D & -\frac{1}{L_1}D^{-1} & C_1D+C_2D+\frac{1}{L_2}D^{-1} \end{bmatrix}$$

The same rules established above for the conductance element network may also be used here except that inductance and capacitance elements must be entered in the form above, using the D operator.

2.2 Loop or path analysis:

A second method of obtaining system equations makes use of the path condition: the sum of the A variables around any closed path is zero. To illustrate this process, we will use the electric circuit below. Once again, we have included only resistors and one type of source to make the illustration simpler. The method is not restricted to systems with only these elements.

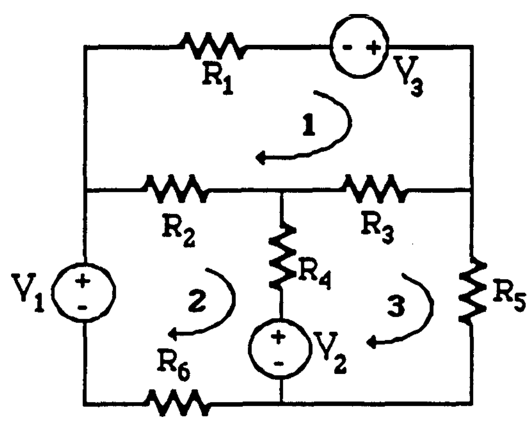


Figure 7 Loop method used to solve a system with 9 branches and. 7 nodes

The system has 9 branches and 7 nodes, so the number of independent loop equations we can write is 8 - 6 = 3. We choose as link currents i1 i2, and i3 which are the currents in the outer paths. (The directions you choose for these currents are completely arbitrary.) We can then use the vertex law at each outer node to express the curents in the inner paths (through R2, R3, and R4) in terms of the link currents. That is, based on the directions chosen above for i1 and i3, we can say that the current in R3 must be i1 - i3 to the left (or i3 - i1 to the right). Similarly, we can establish the other two inner currents. Now notice that if the original link currents were chosen as complete loops, the respective currents flowing in R2, R3, and R4 could be determined by algebraic addition of their loop currents. The following loop equations result:

V3 = i1R1 + (11 -12) R2 + i1 -13) R3	Path 1
Vl-V2=(i2-il)R2+(i2-i3)R4+i2R6	Path 2
V2 = (i3 - i2) R4 + (i3 - il) R3 + i3 R5	Path 3

Rearrange these terms to collect all 1 i2, i3 terms together. .

-V3 = i1 (RI + R2 + R3) - i2 (R2) - i3(R3)	Path 1
VI - V2 = - i l(R2) + i2 (R2 + R4 + R6) - i3 R4	Path 2
V2 = - il(R3) - i2 (R4) + i3 (R3 + R4 + R5)	Path 3

Converting to matrix form:

$$\begin{bmatrix} V_I \\ V_{II} \\ V_{III} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

where V stands for the sum of all voltage sources around the loop specified by the Roman numeral subscript, Rij is the sum of resistances common to paths i and j, and Rii is the sum of all resistances around path i.

The rule for signs is somewhat different with loop methods. **Every Rii** term (main diagonal) is +. The sign of an Rij term is + if paths i and j are chosen to have the same direction through Rij. If paths i and j are in opposite directions through Rij, the sign on the mutual resistance term is -.

If energy storage elements are present, we can use the D and D^{-1} operators as defined above in the discussion of node equations. Figure 8 shows a system containing R, L, and C elements. Loop equations can again be written. The R value of each resistor is again used directly in the matrix. However, now we must use LD for each inductor, and $(1/C)D^{-1}$ for each capacitor.

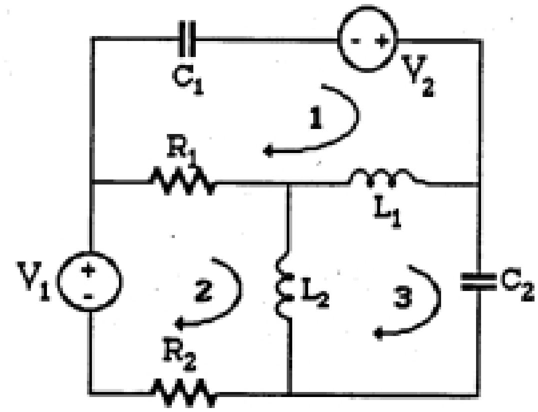


Figure 8 Loop method applied to a system containing R, L, and C

R1 + LID + -L/ C1	-RI	-LID
-R1	R1+ R2 +L2D	-L2D
-L1D	-L2D	LID + L2D + 1/c2 D-1

One restriction remains on our use of loop or nodal methods. We have shown only A-type sources used with loop methods and T-type sources used with node methods. When the "wrong" type source is present, we have two choices. It is possible to convert an A-type source to an . equivalent T-type source and vice-versa, so that the system can always be redrawn with sources of only one type. It is also possible to apply the loop or nodal techniques to systems containing either source type if certain procedures are followed. We will now discuss each of these approaches.

3. Network Theorems

(We illustrate with electrical system elements. Results apply to all linear systems.)

3.1 Thevenin's Theorem

Consider the linear system of the top panel of Figure 9 (box designated LAN: linear active network) which contains assorted sources and passive elements, and has a single "port" by which it can be connected to the outside world. We attach a load, RL, at the port, and observe a current, i1 and a voltage, v1 appearing across that load. (The load can be any element or combination of elements. To make this development easier to follow, we will consider the load to be a resistor.)

In the center panel of Figure 9, a voltage source has been added in series with the load so as to oppose the current flow. We have adjusted the voltage of this source so that the current is zero. When zero current is flowing at a port, we say the port is "open circuited". Therefore, we will call the port voltage of the center panel the *open circuit voltage* of the LAN, Voc This must also be the value of our added voltage source in order to satisfy the path law in the external loop.

In the bottom panel, all independent sources inside the LAN have been turned to zero, creating a linear passive network, LPN. This means that voltage sources have been replaced by short circuits (zero voltage) and current sources have been replaced by open circuits (zero current).

The only source now active is the external source of the center panel, Voc Since this is the only active source, we now expect the current to flow in the direction shown in the bottom panel.

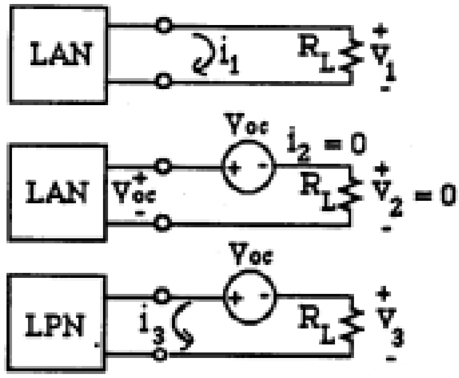


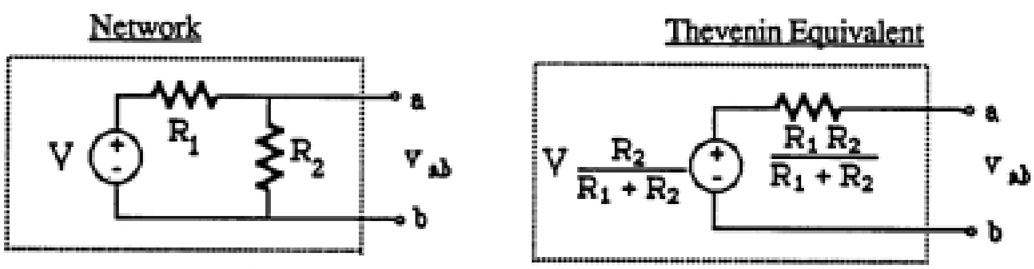
Figure 9 Development of Thevenin's Theorem

According to Thevenin's theorem (stated formally below), the load should receive the same current in the bottom panel configuration as it did in the top panel. To prove this, we apply the superposition principle. In the top panel, with the LAN sources active but the Voc source zero, current i1 *flows* in the load. In the bottom panel, with the LAN sources turned to zero and the Voc . source active, the load current is i3. Now, according to the superposition principle, the current *flow* when all sources are active should be the sum of the currents produced by each acting alone. But this sum is zero according to the center panel. Therefore, i1 and i3, must be equal in magnitude, but opposite in direction. If we reverse Voc in the third panel, the load receives exactly the same voltage and current as when it was connected to the original system.

Statement of Thevenin's Theorem for electric systems:

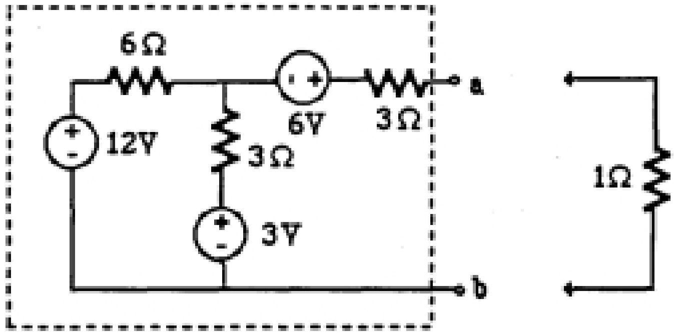
Any linear active one-port network can be replaced by a single voltage source, equal to the open circuit voltage of the one-port, in series with the network in which all <u>independent</u> sources have been set to zero.

Example:

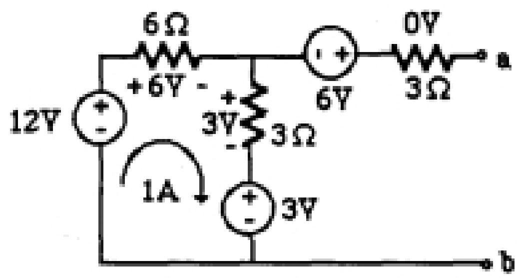


The open circuit voltage of the network on the left can be obtained using the voltage divider formula. This is the voltage source value in the Thevenin equivalent on the right. If we turn source V (on the left) to zero, the resistance seen looking into terminals a-b is the parallel combination of the two resistors. This is used as the Thevenin equivalent resistance in the circuit on the right. The two circuits above are equivalent in that if one is replaced by the other, connected to a system at terminals a-b, there will be no change in any variables of the system. (To verify this, calculate the current each would supply to a load resistor, R3, connected at a-b.

Example:



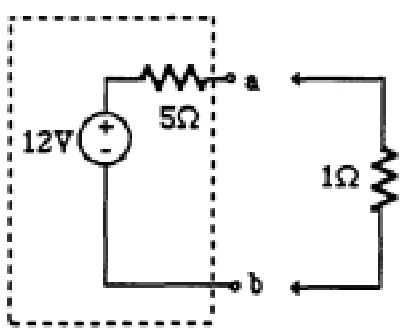
The 1Ω resistor is connected the circuit on the left at terminals a-b. We want to find the voltage across the 1Ω resistor. The Thevenin equivalent may be used to simplify the solution. Disconnect the resistor and find the Thevenin equivalent at terminals a-b. As shown below, with the 1Ω disconnected, the circuit on the left has only one closed loop.



Current in this loop flows clockwise and is (12-3)/'9 =1 amp. To find the voltage across a-b, start at b and travel any path to a and add up voltage rises. For example, for the path through through the 12V source: 12 - 6 + 6 =12V. (Note that the 6V source still contributes to this voltage even though it is in an open path.) Or we could take the 3V path: 3 + 3 + 6 =12V. So the voltage is 12V.

To find the Thevenin resistance, turn all independent sources to zero. The three resistors may be combined to look like 5Ω from terminals a-b. The Thevenin equivalent circuit is therefore a 12 volt source in series with a 5Ω resistor. This may now be reconnected to the 1Ω resistor after which, the voltage divider formula may be used to find the voltage across the 1Ω resistor.

$$12\left[\frac{1}{5+1}\right] = 2V$$



3.2 Norton's Theorem

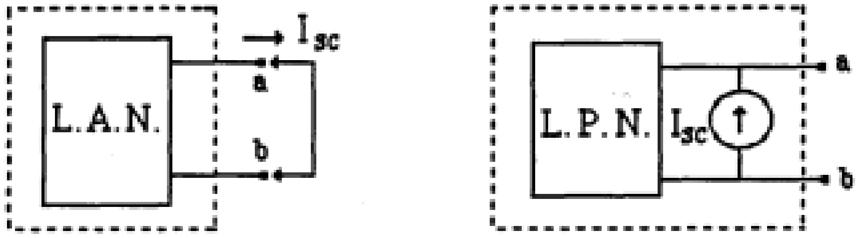
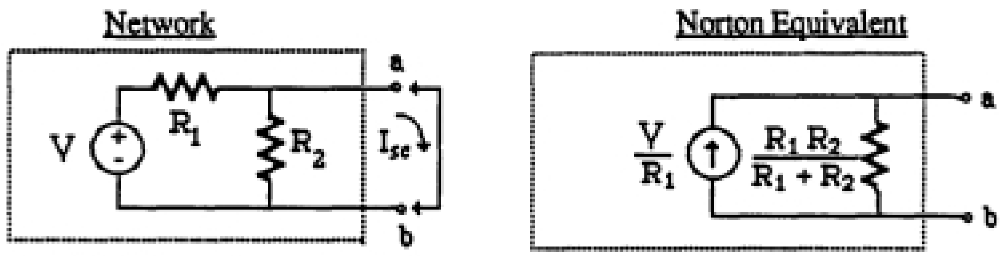


Figure 11 Development of Norton's Theorem

In a very similar development, we can prove that any linear active one-port can be replaced by a current source in parallel with the network with all its independent sources turned to zero. This provides a second way to obtain an equivalent system. The short circuit current can be found by placing a short circuit across the LAN terminals and measuring the current flowing.

Statement of Norton's Theorem for electric systems:
Any linear active one-port network can be replaced by a single current source, equal to the short circuit current of the one-port, in parallel with the network in which all <u>independent</u> sources have been set to zero.

<u>Exile:</u>



We want the Norton equivalent of the network on the left between terminals a-b. Apply a short across a-b. The current in the short, Isc = V/R1 (no current takes the R2 path because of the short). So the Norton current source is V/R1 When the independent source is turned to zero, the network on the left has parallel resistances R1 and R2 between a-b. This is the Norton resistance on the right. .

An example we have seen before which now can be described in terms of Thevenin and Norton equivalents is the simple source transformation given in Figure 10, below. The conversion can go in either direction. The circuit on the right is the Norton equivalent of the one on the left. The circuit on the left is the Thevenin equivalent of the one on the right. The transformed source circuit is equivalent to the original in that it supplies the same output current and voltage to any load connected to it at its terminals. However, neither the resistor of the transformed circuit nor the new source is equivalent to the original. For example, note that under open circuit conditions, the voltage source equivalent has no current in R, but the current source equivalent has I flowing in R. This conversion can be used to convert to all voltage sources or all current sources in preparation for loop or nodal solution.

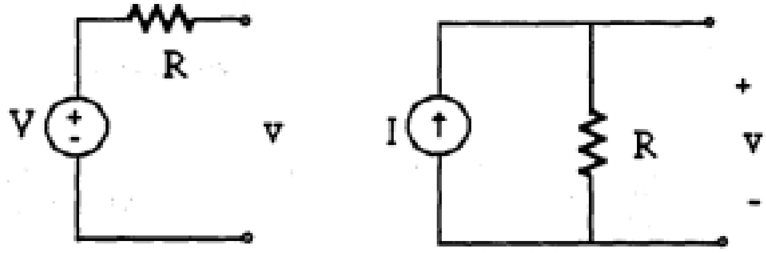


Figure 10 Source Transformations are Thevenin-Norton Equivalents.

3.3 Maximum Power Transfer Theorem

Suppose we connect a resistor, R_L, to the output of a linear network as shown in Figure 11. If $R_L = 0$ (shorting the terminals), no power will be drawn from the network ($i^2R_L = i^2\,0$). If the resistor is infinite (terminals are open) again, no power will be drawn ($v^2/R_L = 0/R_L$). If power out of the network is zero for $R_L = 0$ and again zero for $R_L \to \infty$, there must be a value of R_L between 0 and ∞ for which a maximum power is drawn (proved by Rolle in 1691). What value of R_L draws maximum power?

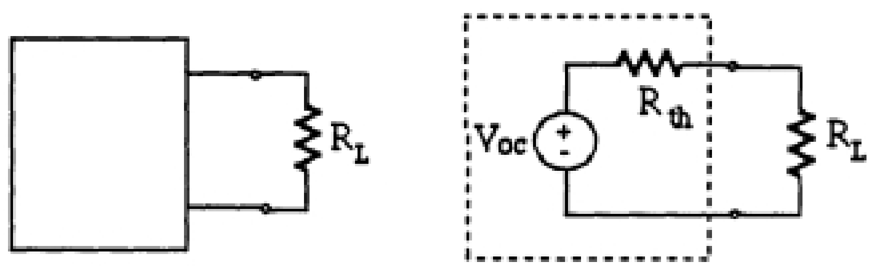


Figure 11 RL draws maximum power when it is set equal to Rth.

To answer this question, replace the general network in the left panel of of Figure 11 with its Thevenin Equivalent, shown in the right panel. The current flowing to the load resistor is $V_{oc}/(R_{th} + R_L)$. Power is I^2R_L so,

$$P = \frac{V_{oc}^2}{[R_{th} + R_L]^2}\,R_L \tag{4}$$

To find the value of RL which draws maximum power, differentiate P with respect to RL and set equal to zero:

$$\frac{dP}{dR_L} = V_{oc}^2\,\frac{[R_{th} + R_L]^2 - R_L\,2\,[R_{th} + R_L]}{[R_{th} + R_L]^4} = 0 \tag{5}$$

Solution of Equation 2 for RL yeilds the condition for maximum power transfer network:

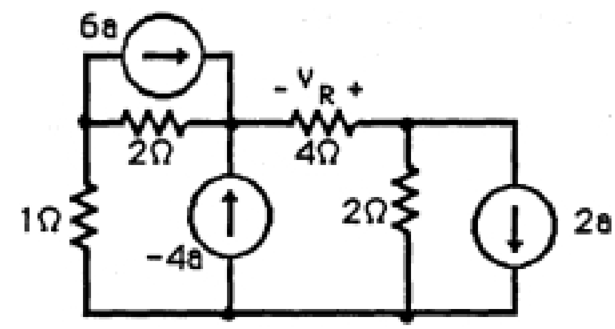
$$R_L = R_{th} \text{ for maximum power transfer.} \qquad (6)$$

Equation 3 implies that even when maximum power is being drawn from a practical energy source, it represents only half of the power generated internally. The value of the maximum available power from any system can be found with reference to Figure 11 with $R_L = R_{th}$:

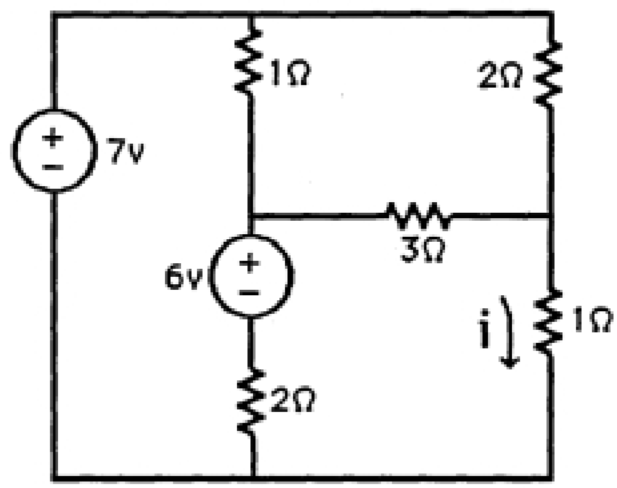
$$P_{max} = i^2 R_L = \frac{V_{oc}^2}{[2 R_L]^2} R_L = \frac{V_{oc}^2}{4 R_L}$$

Chapter 7 Problems

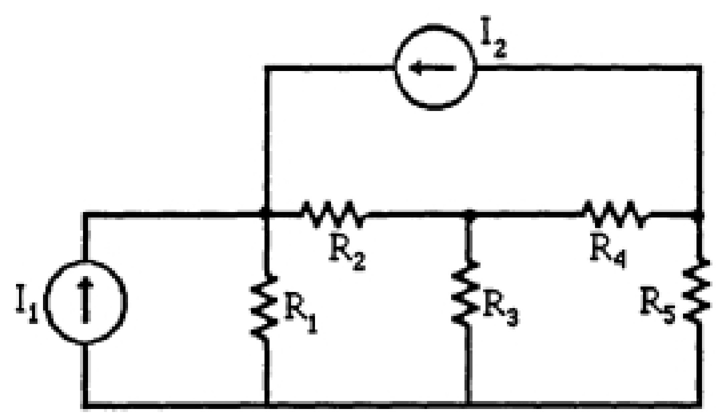
1. Solve for voltage V_R by nodal methods. (Answer. V_R=-36/9 volts)



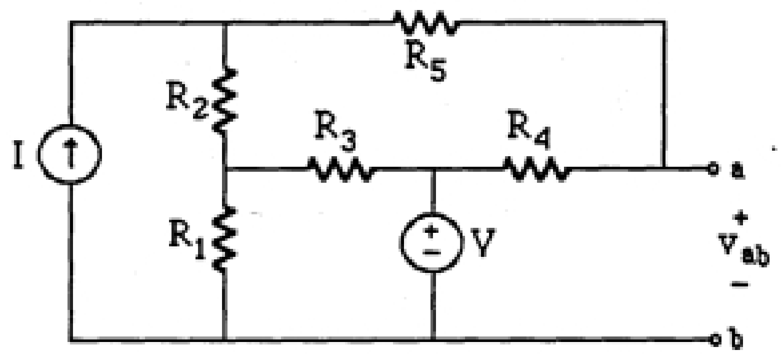
2. Solve for current i by loop methods. (Answer: i = 3 amps)



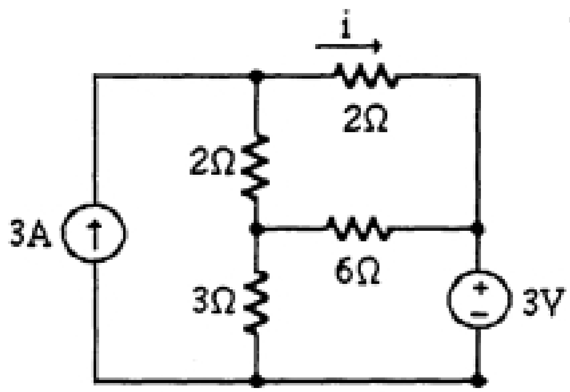
3. (a) Write a set of nodal (vertex law) equations sufficient to solve for the node voltages.
   (b) Write a set of loop (path law) equations sufficient to solve for the loop currents.
   (c) If all resistors are 1Ω, I_1 = 4a and I_2 = 1a, find the voltage across R_5.



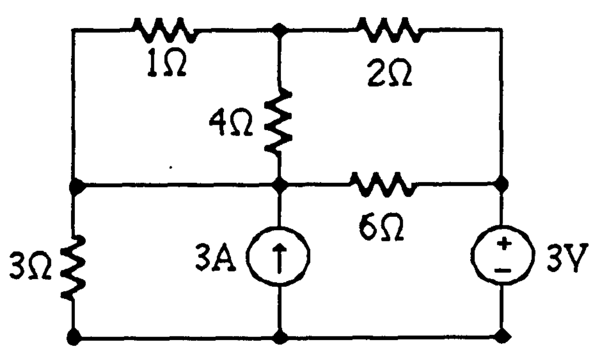
4. (a) Using loop methods, write a set of equations sufficient to solve the circuit below for vab.
(b) Using node methods, write a set of equations sufficient to solve the circuit below for vab. (vab must appear somewhere in the equations.)



5. Find the current labeled i in the circuit below using (a) loop and (b) node methods.

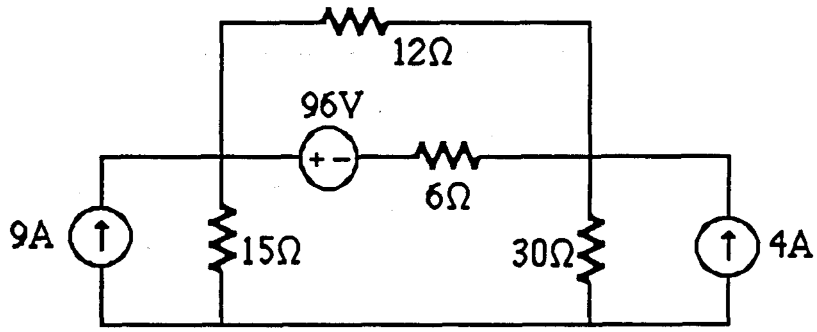


6. Find the voltage across the 3Ω resistor in the circuit below.



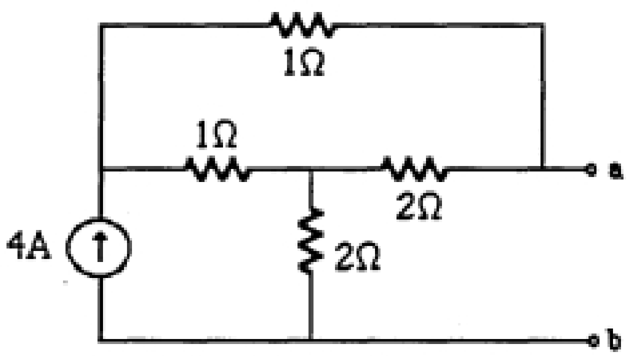
7. (a) Find the current in the 15Ω resistor.

(b) To what value should the 15Ω resistor be changed to draw maximum power?



8. (a) Find the Thevenin equivalent of the circuit below between terminals a-b.

(b) A resistor, RL, is connected at terminals a-b. What value of RL will draw the maximum power from the circuit at terminals a-b? What power will it draw?

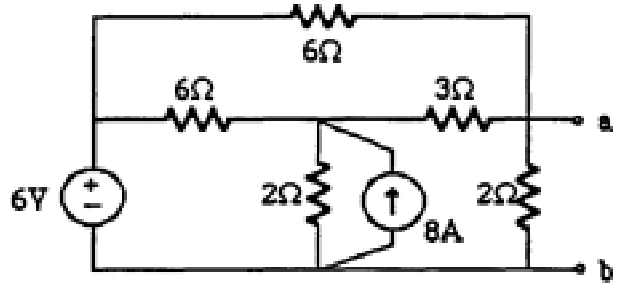


9. For the circuit shown below,

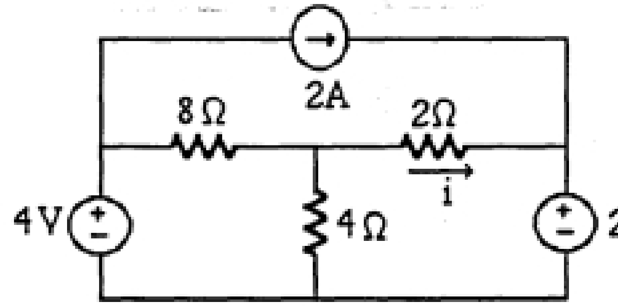
a. Find the Thevenin and Norton equivalent circuits between terminals a-b.
(Ans.: Thev: V_{OC}=9/2V, R_{eq} = 9/8 Ω)

b. What resistor, connected to terminal a-b will draw maximum power from this circuit?
(Ans: 9/8 Ω)

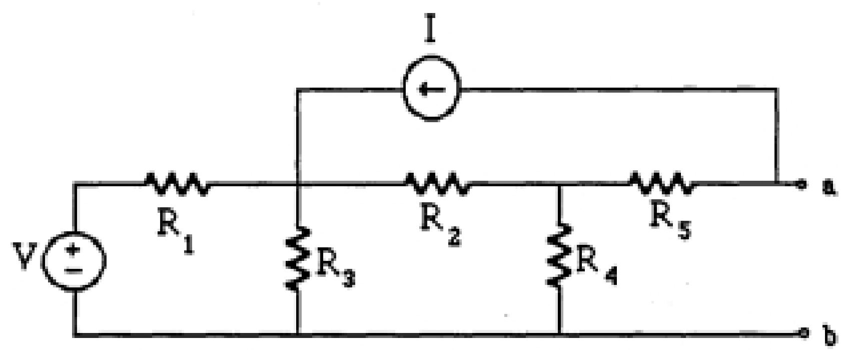
c. What is the maximum power this network can deliver to the resistor of part b?
(Ans: 9/2 W)



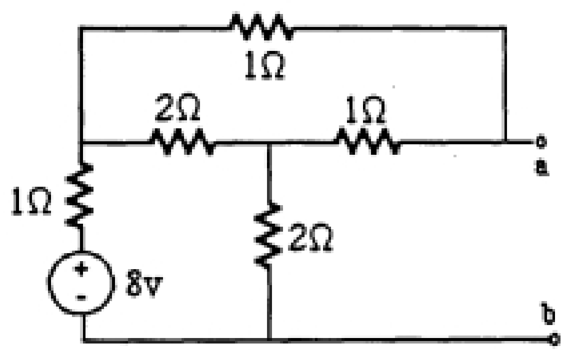
10. Find the value of the current, i, flowing in the 2Ω resistor of the circuit below.



11. Write a set of equations sufficient to solve for vab, the voltage across terminals a-b in the circuit below. To be capable of solving for vab, this variable must appear in at least one equation.

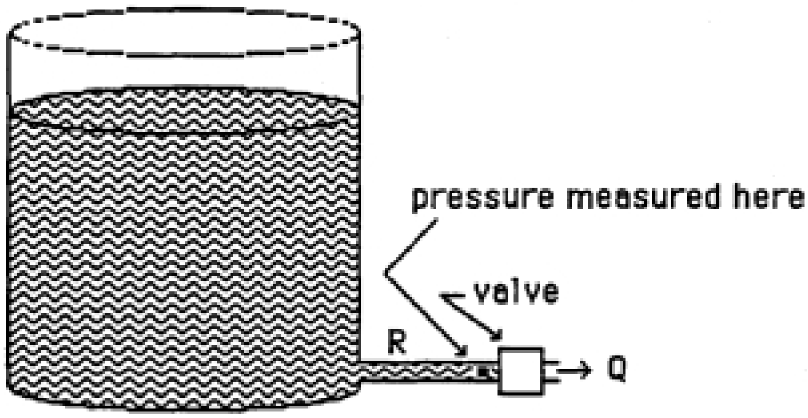


12. Find the Norton equivalent circuit between terminals ab.



13. A source of water flow is the reservoir-pipe system shown below. The pipe carrying water from the pressure source to the valve has resistance. Measurements made at the outlet end of the pipe show the following: with valve V closed, pressure inside the pipe is Po. With valve open and steady flow Q1 coming from the pipe, pressure inside the pipe near the valve falls to P1.

Assume the system is linear and that the reservoir height is not affected. Determine two equivalent models (one using an A source and one using a T source) for this source.



14. Power is transmitted to your house from your own personal electric generator 1 mile away. You measure 120 volts at the wall socket when no appliances are connected. That voltage drops to 118.8 volts when a 50Ω resistor is plugged in. An electric heater is to be constructed for your house. What would be the best resistance for the heating element?

Systems Chanter 8 Study Guide

Zero and First Order System Solutions

A. Concepts Addressed By This Topic

1. Review of mathematical functions used to represent system signals.
   Exponentials, sinusoids.
2. Natural and forced response of first-order systems.
   Time constant, impulse and step response, initial conditions.
3. t = 0+ conditions for A- and T-type elements.

B. Introduction

We have derived mathematical relationships between the across and through variables of ideal, lumped elements of common system types. We have also developed ways of obtaining differential equations of system behavior when the system is represented as the interconnection of several of these ideal elements. We are now ready to look at methods of solution of these system equations.

The differential equation or set of equations we have obtained equate terms containing derivatives of a designated system variable y(t) (called the output or the response of the system), to terms containing an independent forcing function, x(t). In the next two chapters, we will study the solution of equations resulting from zero, first and second order systems in detail.

The solution of a linear system equation is an expression for the *complete* response, y(t). This expression can usually be separated into two parts, found separately. One part, the natural *response*, yn(t), results from the system's internal structure and is independent of the forcing function. It may be determined by setting x(t) to zero. The system equation with x(t) = 0 is *homogeneous* in that all terms contain y(t) or a derivative of y(t). For first order systems, the natural response usually exhibits an exponential decay. The forced response, yf(t), is a function
which satisfies the system equation when a particular x(t) (non-zero) is present. Because of linearity, the complete solution must be the sum: y(t) = yn(t) + y f(t) since this is the sum of the responses to the two sources, x(t) and 0, acting together.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Define time constant for exponential functions. Find the time constant given a graph or expression containing an exponential function'
2. Determine the t = 0+ value of the A or T variable of A- or T-type elements after a switching action.
3. Solve for the natural and forced response of first order systems to singularity function driving forces.

D. Study Procedure

Read Chapter 8 of these notes.

Additional material can be found in Reference 11, Sections 12-1 through 12-3. Also Sections 20.1 through 20.6 of Thomas & Finney calculus text.

Chapter 8
Exponential Function and
Zero and First Order Systems

Exponential Function

The solution of linear system equations almost always includes exponential functions time of the form e^{st}. Before we study procedures for solving the equations of linear systems, we will take some time to investigate these functions which commonly occur whenever a quantity is increased or decreased at a rate which is proportional to the existing amount of the quatity. One example is the growth of principal on an investment.

Suppose you invest some money, P, at 7% simple annual interest. This means that after one year, you have (1+.07)P. If you leave all this money in the bank for the following year you will have (1+.07)(1+.07)P. So after n years you will accumulate $(1+.07)^n$ P. A general formula for the multiplier of your original investment is:

M = (1 + interest rate per period)total number of compounding periods (1)

Although interest rates are specified on a yearly basis, some banks compound payments in a shorter time period, for example, monthly or daily. In these cases, based on Equation 1, your investment is multiplied by:

monthly: $(1 + I/12)^{12Y}$

daily: $(1 + I/365)^{365Y}$

Here, I is the yearly interest rate so we must divide by 12 or 365 to get the interest rate returned in each compounding period. Y is the number of years so we must multiply Y by 12 or 365 to get the total number of compounding periods.

Define h as the interest rate per compounding period:

h =interest rate per year/number of compound in periods in a year

The numerator of (2) is I. Therefore,

Number of compounding periods in a year (3)

So the total number of compounding periods is IY/h. Use this quantity as the exponent of Eauation 1:

$$M = [1 + h]^{\frac{IY}{h}} \tag{4}$$

In general, the more rapid the compounding (smaller h), the faster your money multiplies (larger M). In order to lure investors, some lending institutions offer "continuous" compounding. By this they mean that the compounding period is to be shortened not to hourly or to a second, but to approach zero! What will the multiplier be then?

$$M_{\infty} = \lim_{h \to 0} \left[(1+h)^{\frac{1}{h}}\right]^{IY} \qquad (5)$$

Equation 5 gives the multiplier for continuous compounding. The IY part of the exponent has been taken outside the brackets since I and Y do not affect the limit. Now any good calculus book (or Maple's limit function) will show you that

$$\lim_{h \to 0} \left[(1+h)^{\frac{1}{h}}\right] = e \qquad (6)$$

where e is the base of the natural logarithm, 2.71828182..., an irrational number. So continuous compounding means your money is multiplied by the factor e^{IY}.

The above development illustrates just one instance of the many appearances of the constant, e, in ordinary calculations. The exponential e^{st} will be occurring over and over in the solution of system equations (s is a constant and t is time). For this reason, we will pause here to discuss this function in some detail. Remember that e is approximately equal to 2.718. Examples for various s values are given in Figure 1, below:

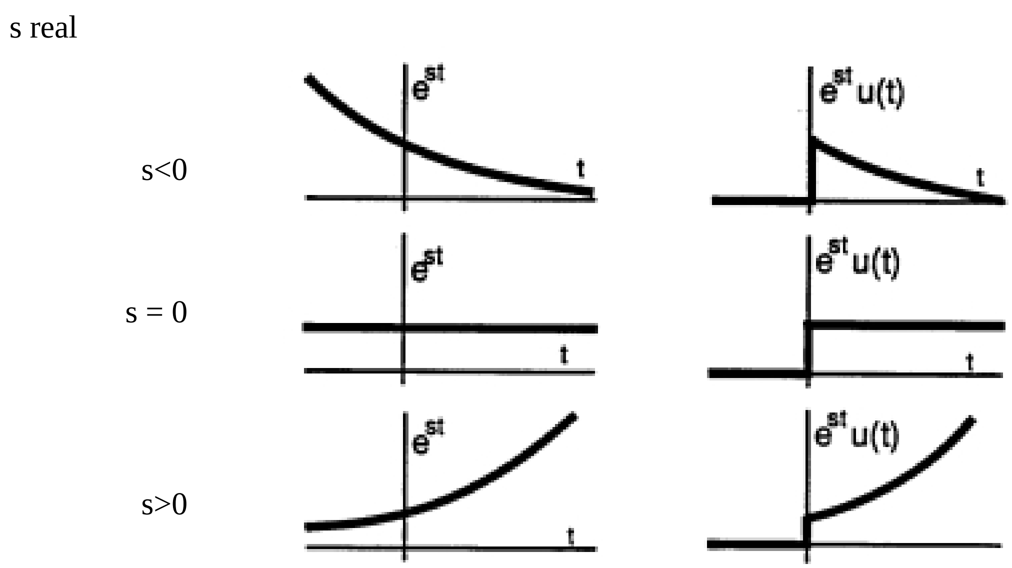


Figure 1 Waveforms associated with real values of s

Another form of the exponential which occurs often is $A[1 - e^{st}]$, with s negative real. It is plotted in Figure 2. Note that all of these functions can be made to start at t=0 by multiplication by a step function.

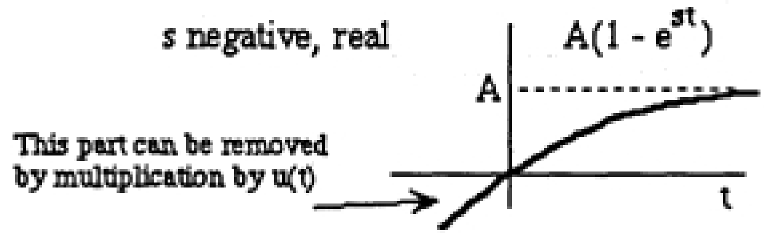


Figure 2 The function $A(1 - e^{st})$ for $s < 0$.

Time Constant

Figure 3 plots the function $K\,e^{st}\,u(t)$, for s real and negative. Define a new constant, $\tau = -1/s$ so that the exponential function can be written as shown in Figure 3. τ is a positive number called the *time constant*. At $t = 0$, the function jumps from 0 to value K because of the step function. It then decays back toward zero. When $t = \tau$, the value is down to $K\,e^{-1}$,which is approximately 0.368 of its initial ($t = 0^+$) value, K. So the function decreases to about 37% of its initial value in one time constant.

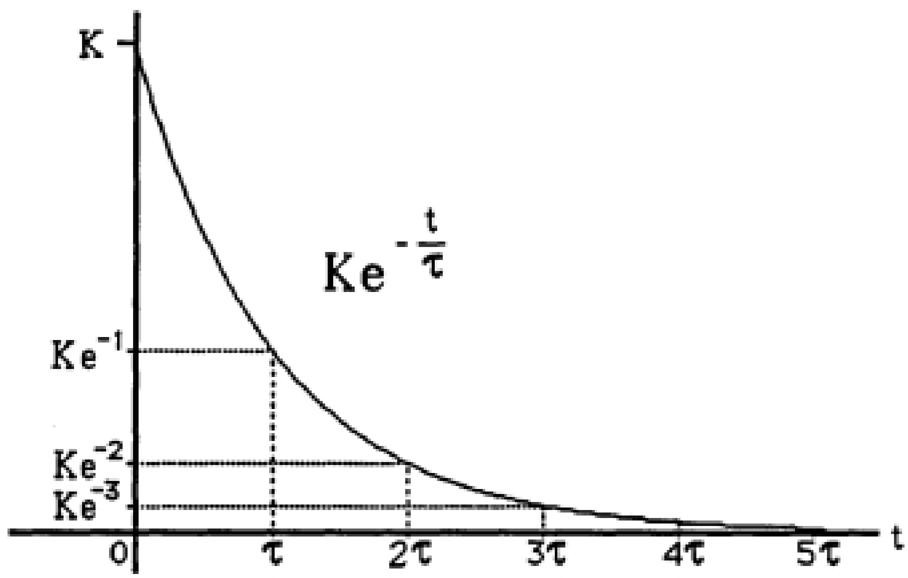


Figure 3 Exponential Decay showing time constants

It is a property of exponential decays, that equal per cent changes occur over equal time periods, regardless of where you start measuring. This means that a new 37% decrease (from Ke-1) occurs from r to 2T, and then from 2i to R, etc. A table of values at each integer time constant multiple shows that the decay can probably be considered complete (for practical purposes) after 5 time constants.

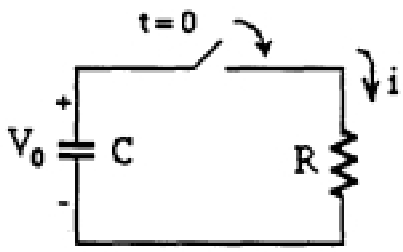
<u>Table 1</u> Fraction ofinitial value remaining after each time constant

elapsed time	$e^{-t/\tau}$
τ	.368
2τ	.135
3τ	.050
4τ	.0183
5τ	.0067

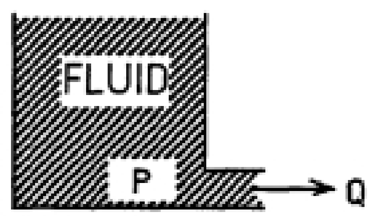
Examples: $5e^{-2000t}$ u(t) decays in approximately 2 ms; $10e^{-0.1t}$ u(t) requires approximately 50s to disappear. (Both look exactly like the curve shown in Figure 3.)

Exponential decaying functions result when the quantity being measured is diminishing its own source. Examples are given in Figure 4. On the left, a charged capacitor discharges through a resistor. On the right, a full reservoir discharges through a narrow pipe.

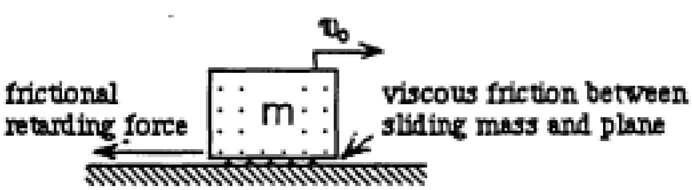
Current i is proportional to voltage which diminishes as current flows:



Flow Q is proportional to pressure, P which diminishes as volume (head) decreases.



Retarding force is proportional to velocity which diminishes as force acts.



ω proportional to torque in damper. Torque diminishes as spring unwinds.

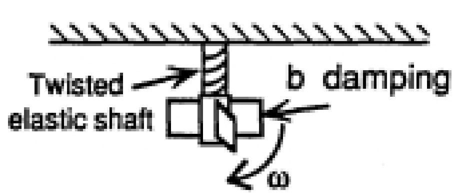
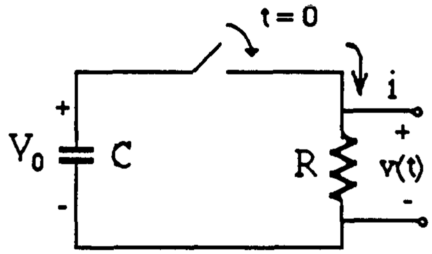


Figure 4 First order systems in which across and through variables both exhibit exponential decay

All the systems of Figure 4 have only one energy storage element (either A or T-type) and one D-type element. This makes them all first-order systems. Note that none of these systems is driven by an external forcing function (x(t) = 0). This means that only natural behavior will be evident.

Example: First-order system driven by a zero forcing function:

A capacitor is charged to Vo volts, then attached to a resistor at t = 0. Find the voltage across R, v(t), for all time.



Solution:

Before the switch closes, the resistor voltage is zero. After the switch closes, the vertex condition at the top node says that currents down through C and R must add to zero.

$$C\frac{dv}{dt} + \frac{1}{R}v = 0 \quad \text{or} \quad RC\frac{dv}{dt} + v = 0$$

We can solve this equation using separation of variables. (We will do this work only once. The result will be used in all subsequent first order cases.)

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrate both sides:

$$\int_{v(0)}^{v(t)} \frac{dv}{v} = -\frac{1}{RC}\int_0^t dt$$

Inserting limits produces ln of a ratio:

$$\ln\frac{v(t)}{v(0)} = -\frac{1}{RC}t$$

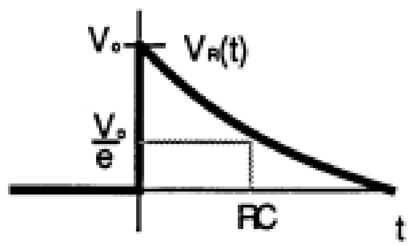
Take the antilog and multiply by v(0):

$$v(t) = v(0)\,e^{-\frac{1}{RC}t}\,u(t)$$

The result is multiplied by u(t) because as the problem was defined, we know the solution above does not apply for t<0 (the switch was open). We also know that the resistor voltage was zero for t<0 so we force this with the step function. What is v(0)? The resistor is experiencing a voltage jump at this instant from 0 to Vo so the value of v(0) is not clear. What we need is the voltage the transition because this is the value from which the decay will take place' We know that since the capacitor is an A-type element, its voltage cannot change instantaneously unless an impulse current is present. This is ruled out by the resistor. Since the capacitor was charged to Vo volts at t = 0- (just before connection), it must still have this voltage, V0, at t = 0+

(just after connection). So v(0) in the solution above, is V0.

The time constant of the exponential decay is $\tau = RC$. The time it takes for the resistor voltage to be negligible compared to its initial value is then approximately 5RC. For example, if C is 1 microfarad and R is 100,000Ω, it will take approximately 0.5s for the decay to be "complete".



Generalization to all First Order Systems:

If the equation for the unknown (v) is written so that it appears with a unity coefficient,

$$RC\frac{dv}{dt} + v = 0 \tag{8}$$

the time constant appears as the multiplier of the first derivative' In a more general form, this equation is

$$\tau\frac{dy}{dt} + y = 0 \tag{9}$$

where y stands for the unknown variable and τ stands for the time constant. As we have already determined, equations such as this have as a solution,

$$y = Ke^{-\frac{t}{\tau}} \text{ for } t \geq 0 \tag{10}$$

<u>Natural Response</u>

This solution is called the <u>natural response</u> (designated vn) of the system because the forcing function (right-hand side of the equation) is zero. The response to a zero forcing function depends (except for K) only on system parameters.

It is important to realize that the across and through variables of all elements in the system also follow this function. For example, current magnitudes in R and in Care both equal to

$$\frac{V_0}{R} e^{-\frac{t}{RC}} \tag{11}$$

First-Order Systems with Forcing Functions Applied

Of course, most systems do not simply react to stored energy, but instead are driven by specific A or T type sources called inputs. We often are interested in a particular variable of the system which is called the system's output. When a source is present, the output response will have two parts. One part, called the forced response, will have the form of the forcing function. A second part, the natural response, will be determined by system elements as in the above example. We will illustrate this using the same RC system above.

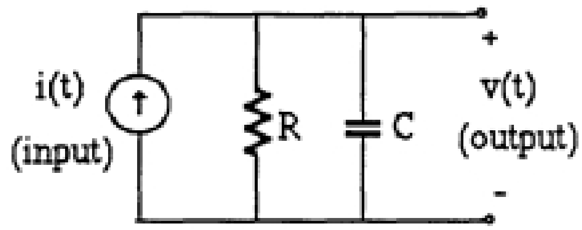


Figure 5 RC System with Forcing Function

The equation relating input and output variables is

$$i(t) = C\frac{dv}{dt} + \frac{1}{R} \qquad (12)$$

where i(t) is the forcing function (input) and v(t) is designated as the output.

Suppose i(t) is zero. Can there be an output? Yes. We found above that the voltage of the form $Ke^{-t/RC}$ can appear across this system with a zero i(t). (Note that this voltage produces equal and opposite currents in R and C.) This response to a zero forcing function is called the natural response of the system.

Suppose i(t) is non-zero, equal to A u(t). Then for t>0,

$$A = C\frac{dv}{dt} + \frac{1}{R} \qquad (13)$$

A solution of this equation is v equal to a constant, AR. This v is called the forced response, designated v f, because it is the v needed to satisfy the forcing function of the equation.

If we now add $Ke^{-t/RC}$ to v_f, we are adding zero on the right side of the equation, based on our previous example. So to provide a general solution, we must add the natural and forced components: $v = v_n + v_f$:

$$v(t) = v_f + v_n = AR + Ke^{-\frac{1}{RC}t}, \quad t > 0. \qquad (14)$$

We know that at t = 0+, v = Vo so AR + K = V0, or K = Vo -AR.

Then the complete solution for the voltage after the step function driving force is applied is

$$v(t) = AR + (V_0 - AR)\, e^{-\frac{1}{RC}t}, \quad t > 0. \qquad (15)$$

This can be expressed in a general way which fits all first order systems driven by a step function:

$$\text{Response} = \text{Final Value} + (\text{Initial Value} - \text{Final Value})\ e^{-\frac{t}{\tau}}, \quad t > 0. \tag{16}$$

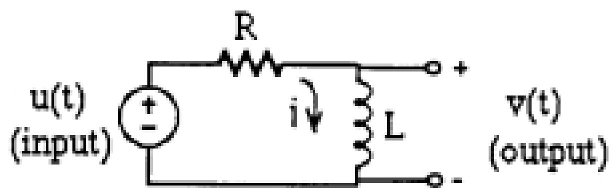
Unit Step Response

A special case of the system responses we are studying here is the unit step response of a system. By definition, this is the response which occurs when the system initially has zero energy storage, and a unit step forcing function is applied at the input' For the earlier RC example, the result has already been found in terms of the input step function magnitude, A. Now, if A=1, the unit step response is

$$\frac{1}{R}\left[1 - e^{-\frac{1}{RC}t}\right] u(t). \tag{17}$$

Example:

For the system below, the across variable of L is defined as the output. Find the unit step response.



Solution:

A path equation gives u(t) = Ri + L di/dt. But we need an equation in terms of v, since this is chosen as the output. We recognize that

$$i = \frac{1}{L}\int v\,dt \tag{18}$$

Substituting this for i gives the equation in terms of

$$u(t) = \frac{R}{L}\int v\,dt + v \tag{19}$$

To obtain the natural part of the response, we need the homogeneous equation (zero forcing function)' Since it is clear that the system is first-order, we know the natural solution for t>O is K $e^{-t/\tau}$. Differentiating Equation 19 when u(t) is replaced by zero gives

$$0 = \frac{dv}{dt} + \frac{R}{L}v \tag{20}$$

We see from this that the time constant, ,r, is L/R. So the natural response

$$v_n = e^{-\frac{R}{L}t} \qquad (21)$$

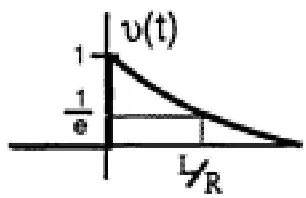
For the forced response, we can consider Equation 19 for t very large. By this time, the natural response has died away and v=vf. We expect of to be a constant because the forced response has the same form as the forcing function. Based on Equation 19, if v f * 0, the integral will continue to grow. Therefore, for this case, the forced response is zero. (Another way to determine this is to realize that in equilibrium, with a constant forcing function, all variables become constants. In this case, Ldi/dt must be zero.

The complete unit step response is then

$$v(t) = v_n + v_f = Ae^{-Rt/L} + 0. \qquad (22)$$

Our remaining job is to determine A. To do this, we use the conditions at t = 0+. In a T-type element such as L, the flow variable cannot change instantaneously unless an impulse appears across the element. Therefore, for this case, the flow in L at t = 0+ must be the same as it was at t = 0-. At t = 0-, the flow was zero (the system was dead)' So the current is zero at t = 0+. If current is zero, no voltage appears across R (vR = iR). Therefore, at t = 0+, the 1 volt step all appears across L' That is, vL(0+) = v(0+) = 1. Then from Equation 22, A =1. Finally, the unit step response of the RL system is

$$v(t) = e^{-\frac{R}{L}t}\ u(t) \qquad (23)$$



We may now formalize a method for finding the response of a first order system to a constant forcing function:

For equations of the form $Au(t) = a_1 \frac{dy}{dt} + a_2 y$:

1. Find the time constant $\tau = \frac{a_1}{a_2}$. Nore: This will always be either $\frac{L}{R}$ or RC.

2. Find the initial (t = 0+) value of y, y(0+). (This will sometimes be given.) Use the IC rule: the variable which determines the energy storage cannot change instantaneously unless an impulse is present. If an impulse is present, the change in the variable is 1/C or 1/L times the size of the impulse.

3. Determine the final value of y, $y(\infty) = A/a_2$. (By assuming dy/dt becomes zero eventually.)

4. Solution is $y(t) = y(\text{final}) + [y(\text{initial}) - y(\text{final})]\, e^{-t/\tau}$.

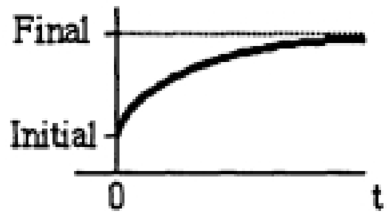
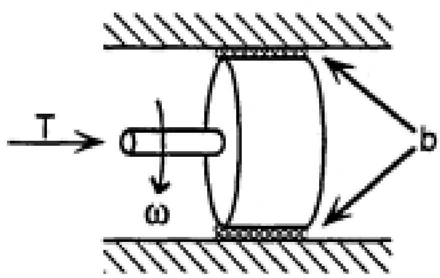
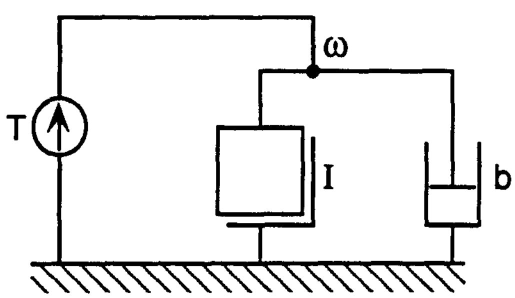


Figure 6 Response of first order system to step change

Examples:
1.The flywheel (moment of inertia = I) shown below is coasting and slowing due to friction and has reached an angular velocity of coo at t=O. The bearing friction may be considered to be ideal viscous damping of coefficient b. At t = 0, a torque, T, is suddenly applied to increase the flywheel's angular velocity. Find w, the angular velocity of the flywheel, for t>0.



Solution: The first step in the solution is to model the system using ideal elements. In this system there is one angular velocity (that of the flywheel), so we know there is only one node in addition to the reference node. There are two ideal elements (flywheel inertia and damping) plus an ideal T-type source (Torque). Draw the two ideal elements between the velocity node and the reference node. The source must also connect between these nodes.



We can now write a node equation for this system:

$$T\,u(t) = I\frac{d\omega}{dt} + b\,\omega$$

l. By considering the homogeneous form of this equation (zero on the left side), we can find the time constant, $\tau = I/b$.

2. At $t = 0^-$ we are told the flywheel had an angular velocity ω_0. Using our continuity rule for A-type energy storage elements, we can say that the angular velocity at $t = 0^+$ must still be ω_0. In other words, $\omega(0^+) = \omega_0$.

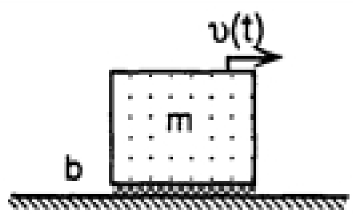
3' The final value of w is T/b. This is obtained by evaluating the system equation for large t when all derivatives will be zero.

4. Using Equation 16, the

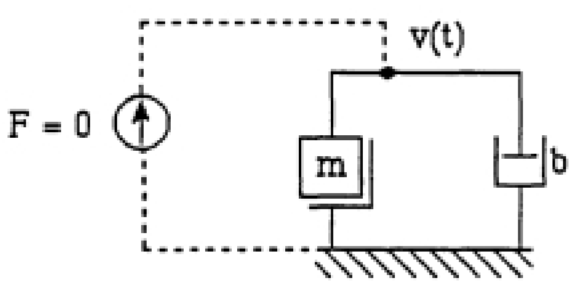
$$\omega(t) = \left(\frac{T}{b} + (\omega - \frac{T}{b}) e^{-\frac{b}{I}t} \right), \; t > 0$$

Notice that the forced part of the response is the same form as the forcing function (first term, a constant for t>0), and that the natural part of the response is the exponential form dictated by the first order system.

2. The mass, m, shown below has a velocity of V0 m/s at t = 0. The viscous friction coefficient is b. Find its velocity for t>0.



Solution: Model the system using ideal elements:



Write a vertex law equation at the top

$$0 = m\frac{dv}{dt} + b\,v$$

Rearrange the constants to get the form of Equation 9:

$$0 = \frac{m}{b}\frac{dv}{dt} + v$$

We recognize the time constant $\tau = m/b$. The solution to this homogeneous equation is

$$v(t) = v_n(t) = V_0 e^{-\frac{b}{m}t}, \quad t > 0.$$

Note that since there is no forcing function (source element), the system exhibits only its natural response' Also notice the statement t > 0, which is necessary because we have no information about the behavior of the system prior to t = 0. In this case, it is not correct to use a step function because this does specify behavior for t<0.

First order„ stems with other than constant forcing functions:

Obviously, there are many possible forcing function waveforms beside constants or steps. In these cases, we cannot always find the forced response by assuming that after a long time all derivatives are zero, as we have done in the examples above. For some cases, such as sinusoidal or exponential forcing functions, special techniques have been devised to help find the forced response. We will study these in upcoming sections of this course. One rule that does always hold is that the forced response always resembles the forcing function. That is, sinusoidal **forcing** functions produce sinusoidal forced response terms. Exponential forcing functions such as e^{-at} produce terms in the forced response which contain e^{-at}, where a is not related to the time constant of the system. On the other hand, the form of the natural response is determined by the system's passive components as we have seen above. For first order systems, it always takes the form $K e^{-t/\tau}$. This form will appear in the solution of any first order system, regardless of the type of forcing function applied to the system, or the initial energy storage specifications. Since only the homogeneous equation is needed to determine τ, this part of the response is independent of the forcing function.

Zero Order Systems

Before we move on to study systems of higher order, we should say a word about zero order systems. Remember the simple rule for determining system order. differentiate or integrate through the system equation until no integrals of either x or y remain, but so that at least one term contains either x or y undifferentiated. System order is then the highest derivative of the dependent variable, y. A zero order system is therefore one whose equation takes one of the forms below:

$$y(t) = K\,x(t)$$

$$y(t) = a\frac{dx}{dt} + bx$$

$$\text{or, in general, } y(t) = \sum_{i=0}^{n} a_i \frac{d^i x}{dt^i}$$

where any of the ai can be zero. So the output of a zero order system is made up of simply the forcing function and/or its derivatives. Therefore the system has no memory. The output at any instant is a function of the input at that instant only. This means a zero order system does not exhibit a natural response, only a forced response.

Chapter 8 Problems

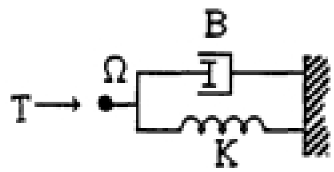
I' Solve each of the following equations for y(t), good for t > 0. Estimate how many seconds are required for each system to reach steady state.:

$$3\,dy/dt + 2y = 0;\ y(0) = 2$$
$$5\,dy/dt - 3y = 0;\ y(0) = 3$$
$$dy/dt + 3y = 6\,u(t);\ y(0) = 0$$

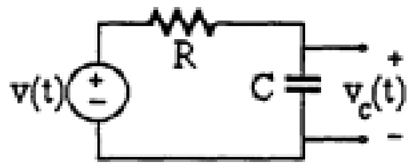
2. For each system below:

(a) Draw a diagram using generalized symbols.

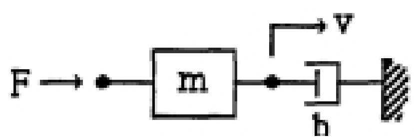
(b) Determine the output if the input function is Au(t) and all systems have no energy storage at $t = 0^-$.



Input T, output Ω



Input v(t), output $v_c(t)$

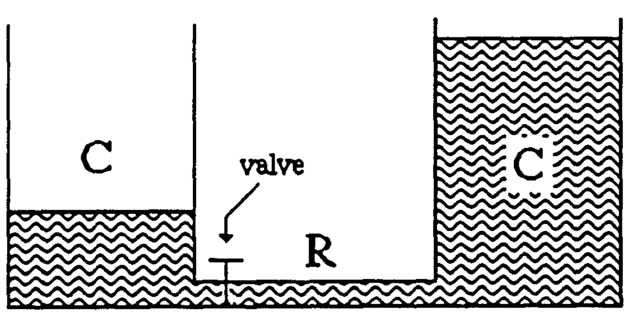


Input F, output v

(c) Repeat (b) if 2J of energy is stored in the energy storage element of each system at $t = 0^-$.

3. Two water tanks are joined by a pipe at their base as shown below. The pipe has a resistance to flow of R. The tanks are of equal size and each introduces a fluid capacitance, C. A valve in the pipe which was closed, is opened at t = 0. Approximately how long will it take for the water levels in the tanks to equalize? Explain how you arrived at your answer.

Fluid resistance of the pipe: $R = 1000\ N{\bullet}s/m^5$
Fluid capacitance of each tank: $C = 0.1\ m^5/N$



Systems Chanter 9 Study Guide

Second Order Systems

A. **Concepts Addressed By This Topic**

1. Review of mathematical functions used to represent system signals.
   Complex numbers, sinusoids.
2. Natural frequencies of second-order systems.
   Ddistinct, repeated, complex roots of the characteristic equation.
3. Special parameters: Damping
   Damping coefficient, damping ratio, undamped natural frequency.
4. Complete response with constant forcing functions.

B. Introduction

The differential equation representing second order systems results in a quadratic characteristic equation. Therefore, we can expect two roots instead of the single root we found for first order systems' We will find that the exponential function again is the form of the natural solution, but now two exponential terms will be required. Furthermore, since the roots of a quadratic equation may be complex conjugates, or repeated, we must investigate the resulting solution these forms produce. We will find that in addition to the exponential decay form found in first order systems, we now have the additional possibility of oscillation in the natural response.

Exponential functions and sinusoidal functions are related by Euler's Identity. This identity leads to a new way (phasors) of handling sums and differences of sinusoids. The exponential multipliers of t which occur in the natural response will be associated with such physical attributes of the response as time of decay and frequency of oscillation. These multipliers are easily obtained from the coefficients of the original equation, making it possible to predict much of the natural behavior of second order systems without a complete analytic solution.

Each forcing function produces a response particular to that function. Certain functions are convenient for studying a system's behavior. These include the unit step, the unit impulse, and the sinusoid, among others. We will determine the unit step and impulse response of second order systems in this chapter.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Define amplitude, frequency, period, and phase angle for sinusoidal functions. Find these quantities given exponential or sinusoidal expressions.
2. Use complex phasors to determine the result of arithmetic combination of sinusoids. Use Euler's identity to convert between complex exponential and sinusoidal expressions.
3. Determine the "complex frequency" of sinusoids with exponentially changing amplitudes' Relate points in the s plane to these frequencies.
4. Find the damping ratio and the undamped natural frequency of second order systems. Determine the level of damping (over, under, critical).
5. Determine the complete response of second order systems to singularity forcing functions.

D. Study Procedure

Read Chapter 9 of these notes.
Additional material can be found in Reference 1, 6, and 8.

Chapter 9
Second Order Systems

The methods we will use to solve second order systems are very similar to those used with first order systems. Some additional complexity will be introduced because the characteristic equation now has two roots (a quadratic), and two initial conditions are required (because there are two energy storage elements). However, the procedure of determining the natural and forced responses and adding them is still valid.

In first order systems, the root of the characteristic equation, s, was always real. With second order systems, the two s roots may be complex conjugates. We will see that complex roots introduce another time function which is very common to linear systems: the sinusoid: In systems of second or higher order, sinusoids are often part of the natural response. A simple mathematical relationship (Euler's Identity) exists between the sinusoidal function and e^{st}.

1.1 Review of Complex Algebra

A complex number is simply a number pair. Unfortunately the names "real part" and "imaginary part" have been adopted as the names of the two parts of the complex number. The names do not mean that either part is more authentic than the other.

Since a complex number has two parts, the most convenient way to represent it graphically is as a point in a plane. We can use the horizontal (real) axis to indicate the first number of the pair (the real part), and the vertical (imaginary) axis to indicate the second number of the pair (the imaginary part). Figure 1 shows how the complex number (5, 3) can be depicted on the complex plane. (Here, italics are used to designate the imaginary part.)

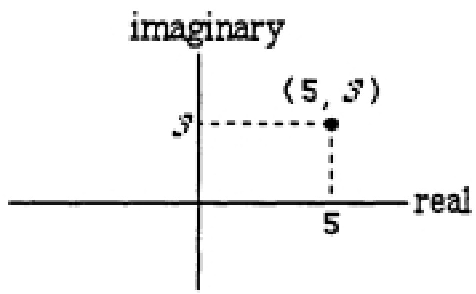


Figure 1 Graphical Representation of a Complex Number

In order to define mathematical operations which involve both parts of complex numbers, we need. to express both parts in terms of one number type. To do this, we define an operator, j, which has the ability to convert a real number to an imaginary number of the same magnitude. j operating on real number, a, converts it to imaginary number, a. That is, j a = a

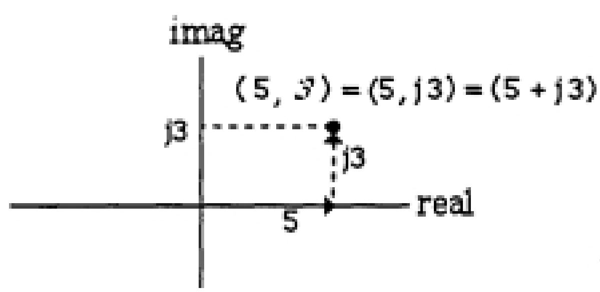


Figure 2 Use of Operator j

So complex number (5, 3) can be expressed (5, j3). Graphically (Figure 2) we see that this is the same as 5 + j3. That is, 5 units in the real direction plus 3 units in the imaginary direction.

The effect of operator j is a rotation of the number operated upon by 900 in the complex plane. Figure 3 shows this effect of operator j , rotating the point at 4 on the real axis to the point 4 , or j4 on the imaginary axis.

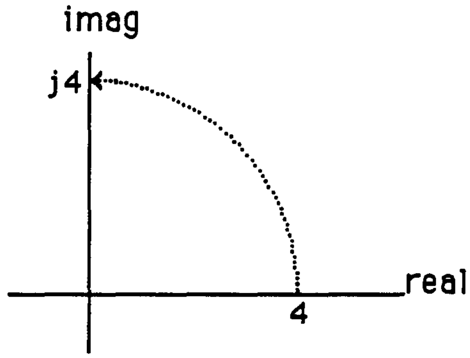


Figure 3 j Operator Rotates Point 90°

Suppose we operate a second time with j:

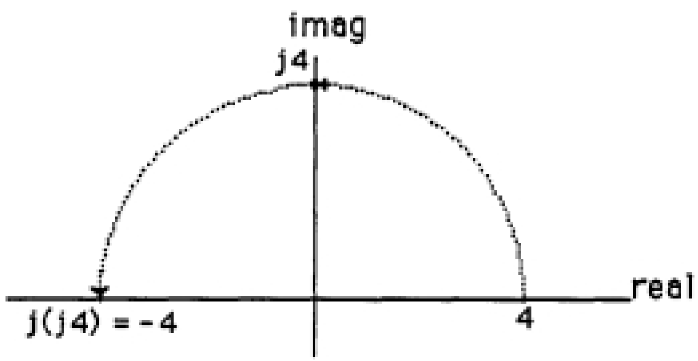


Figure 4 j^2 Equivalent to -1

Figure 4 indicates that j(j4) rotates the point twice 90^o to the negative real axis where it falls on the value -4. So operating with j twice (j^2) is equivalent to multiplication by -1. Even powers of j can therefore be converted to +1 or -1. Odd powers of j can be converted to +j or -j. Some examples: $j^4 = 1$, $j^3 = -j$, $j^{-1} = -j$.

Since use of the j operator permits both the real and imaginary parts of the complex number to be expressed with real numbers, we may now apply all the usual rules of algebra to combinations of these parts. (We will use bold face type to indicate a complex number.) Given that complex number $\mathbf{Z_1}$ = a + jb, and $\mathbf{Z_2}$ = c + jd, then

Z1 +Z2 = a + c + j (b+d)

Z1 -ZZ =a -c+j(b-d)

$$Z_1 Z_2 = ac + jbc + jad + j^2bd = (ac - bd) + j(bc + ad)$$

$$\frac{Z_1}{Z_2} = \frac{a + jb}{c + jd}$$

To separate real and imaginary parts of Z_1/Z_2, multiply by (c - jd)/(c - jd). Note that (c + jd)(c - jd) = $c^2 + d^2$. This process is called rationalizing the denominator.

$$\frac{a + jb}{c + jd}\,\frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

<u>Polar Form</u>

If we use a two-<Hmensional graph to plot complex numbers, it is possible to locate a given point given its real and imaginary parts. A second way of locating the same point is by specifying
a distance from the origin (the magnitude M of the complex number), and a direction (the angle 8 measured from the positive real axis). A shorthand way of designating magnitude M and angle angle θ is $M\angle\theta$

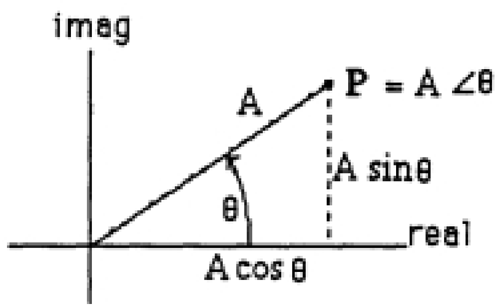


Figure 5 Complex Number in Polar Form

By geometry, we can derive the following relations using Figure 5 (absolute value bars indicate magnitude of the complex number):

$$A = \sqrt{a^2 + b^2} = |Z|$$

$$\theta = \tan^{-1}\frac{b}{a} \qquad (1)$$

Inversely,

$$a = A\cos\theta$$
$$b = A\sin\theta \qquad (2)$$

1.3 **Euler's Identity**

A mathematical relation called Euler's Identity is used extensively in systems and signal analysis. It enables the conversion of harmonic functions between exponential and trigonometric forms. Consider the complex number P =M angle θ of Figure 5:

$$P = A\cos\theta + jA\sin\theta$$

$$\frac{dP}{d\theta} = A\left[-\sin\theta + j\cos\theta\right] = jA\left[\cos\theta + j\sin\theta\right] = j\mathbf{P}$$

$$\frac{d\mathbf{P}}{\mathbf{P}} = j\,d\theta \qquad (4)$$

$$\int \frac{d\mathbf{P}}{\mathbf{P}} = \int j\,d\theta$$

$$\ln \mathbf{P} = j\theta + K \qquad (5)$$

We can evaluate K by choosing any convenient angle for θ (such as 0^{o} or 90^{o}). If $\theta = 0$, $\mathbf{P} = A\cos 0 + jA\sin 0 = A$. Then Equation 5 says that

$$K = \ln A \qquad (6)$$

So, $\ln \mathbf{P} = \ln A + j\theta$

Taking antilogs,

$$\mathbf{P} = e^{(\ln A + j\theta)} = e^{\ln A}\, e^{j\theta} = A\,e^{j\theta}$$

Substituting from Equation 3,

$$\mathbf{P} = A\cos\theta + jA\sin\theta = A\,e^{j\theta} \quad (\text{or } A\angle\theta) \qquad (g)$$

Equation 8 is known as Euler's Identity.

By changing signs on 8 and repeating the above development, we can show that

$$A\cos\theta - jA\sin\theta = A\,e^{-j\theta}$$

Addition and subtraction of Equations 8 and 9 yield other Euler

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad (10)$$

2. Second Order Systems

A second-order system has two independent energy storage elements' Independent in this case means that the A-type variable of any A-type energy storage element, and/or the T-type variable of any T-type energy storage element may be set independently' An example of a nonindependent pair of energy storage elements would be two masses connected together so that they have one common velocity. Another example is given in Figure 6. Two inductor-type elements in series must have the same T variable' Therefore, this configuration does not represent two independent energy storage devices. (Two such inductances could be combined to give a single inductor, L1 + L2.)

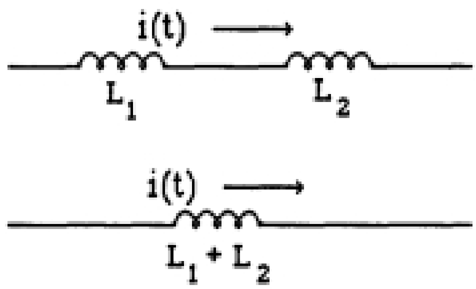


Figure 6 Equivalent Inductances: One independent energy storage element.

When the system has two independent energy storage devices, the resulting second-order equation cannot be solved in exactly the same manner as that used with first-order systems. To illustrate the procedure, we will solve a representative system under several conditions.

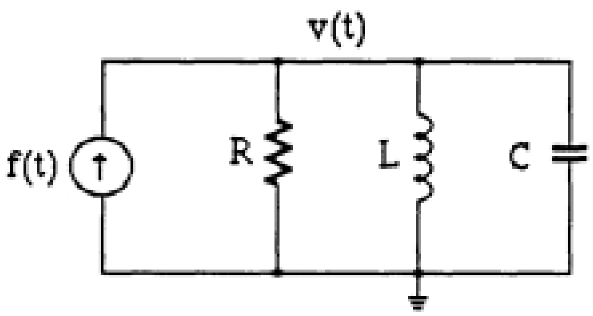


Figure 7 RLC Parallel System

Apply nodal techniques at the top node of the system of Figure 7:

$$f(t) = C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int v\,dt \tag{11}$$

We divide the solution of this equation into two parts. Suppose f(t) is turned to zero. There could still be a non-zero v(t) across the system if it causes the three terms on the right to add to zero. Such a v(t) is called the natural response of the system, vn(t). (It is the solution to the homogeneous differential equation.) Since Vn(t) produces a zero on the right side of Equation 11, there must be an additional part of v(t) which produces a value equal to f(t). This is called the forced response, v f (t). The total response of the system is therefore

$$v(t) = v_f(t). + v_n(t). \qquad (12)$$

The forced response is by definition, determined by the (arbitrary) forcing function. Except for some special functions (constant, sinusoid, exponential) solution of the forced component of the response can be difficult. For now, we will concentrate on finding the natural response, vn (t).

Turn the T-type source to zero in the system:

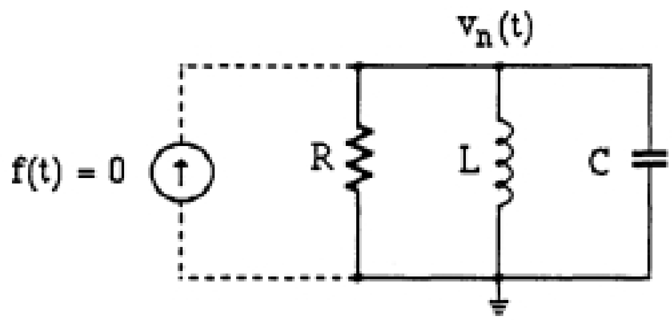


Figure 8 RLC System with Zero Driving Force

The nodal equation is

$$0 = C\frac{dv_n}{dt} + \frac{1}{R}v_n + \frac{1}{L}\int v_n\,dt \qquad (13)$$

vn, signifying natural response is now appropriate because of the zero on the left side of the equation. Differentiating and dividing Equation 13 by C gives:

$$\frac{d^2v_n}{dt^2} + \frac{1}{RC}\frac{dv_n}{dt} + \frac{1}{LC}v_n = 0 \qquad (14)$$

For the first order systems, we found that $v_n = A\,e^{st}$ was the natural solution.. Let's try this solution again. Substitute $v_n = A\,e^{st}$ into Equation 4 and factor out the common $A\,e^{st}$:

$$A\,e^{st}\left[\,s^2 + \frac{1}{RC}s + \frac{1}{LC}\right] = 0 \qquad (15)$$

The values of s which satisfy Equation 5 are the roots of the quantity in the brackets' These can be found using the quadratic formula:

$$s_1 \,,\; s_2 \;=\; -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \tag{16}$$

We assumed the solution was $v_n = A\,e^{st}$, but we have now found two values of s which satisfy Equation 5. Each, if used in the exponent of $A\,e^{st}$ will satisfy the homogeneous equation (4). For the general case, we must allow both terms to be present, so the general form of the natural response is

$$v_n(t) = A_1\, e^{-\,s_1 t} + A_2\, e^{-\,s_2 t}. \tag{17}$$

We illustrate with a numerical example. Suppose in the original system, C = 1/12 F, R = 3 Ω, and L = 4 H as shown in Figure 9.

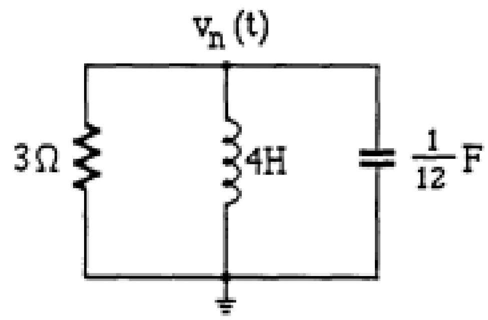


Figure 9 System for Numerical Example

Equation 14 now becomes

$$\frac{d^2 v_n}{dt^2} + 4\frac{dv_n}{dt} + 3\,v_n = 0 \tag{18}$$

Substitution of $v_n = A\,e^{st}$ into Equation 14, and clearing of the e^{st} factor yields the quadratic equation $s^2 + 4s + 3 = 0$ for which the roots are $s_1 = -1$, $s_2 = -3$. (This equation is sometimes called the *characteristic equation*). Then,

$$v_n(t) = A_1\, e^{-\,t} + A_2\, e^{-\,3t}. \tag{19}$$

The constants A 1 and A2 can be found from initial conditions imposed on the system. 'The methods for this determination will be discussed later.

Since s1 and *s2* have the units sec-1, they are called "frequencies". The *natural frequencies,* sn , of this system are -1 and -3. Since either or both of the terms of Equation 19 satisfy Equation 18 for all A values, these terms should be expected as part of v(t) even if a T-type forcing function is driving the system (right side of Equation 11 not zero). For example, if an ideal current source is connected across the system, the voltage may have additional terms (the forced response) but the terms of Equation 13 can also appear.

Equations such as (13) and their solution (17) occur in all branches of engineering systems. For example consider the mechanical system of Figure 10 in which force is the input and velocity is the output of a the mass, spring, damper system.

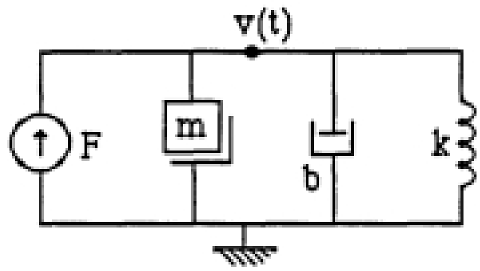


Figure 10 Mass, spring, damper system with force driver.

The equation of this system is

$$F = m\frac{dv}{dt} + b\,v + k\int v\,dt \tag{20}$$

Note that this equation is identical to Equation 11 except for the coefficients' Because the solution of a given second order equation is the same regardless of what type system generated it, certain constants have been defined to make such solutions more uniform. If we obtain the homogeneous form of any second order equation such as Equations 13 and 20, and divide through by the coefficient of the highest derivative, the result is an equation similar to Equation 21.

$$\frac{d^2 v_n}{dt^2} + 2\alpha\frac{dv_n}{dt} + \omega_n^2 v_n = 0 \tag{21}$$

Here, new constants have been defined which apply to all second order systems. The **damping coefficient,** α, is a quantity which determines the rate at which the natural response dies out. The **undamped natural frequency,** ω_n, is related to the frequency of oscillation of systems which exhibit this response. For the systems of Equation 14 (roots given in Equation 16), and Equation 20, these values are:

Damping Coefficient α: $\dfrac{1}{2RC}$ or, $\dfrac{b}{2m}$

Undamped Natural Frequency, ω_n: $\dfrac{1}{\sqrt{LC}}$ or, $\sqrt{\dfrac{k}{m}}$ (22)

For other systems or other configurations of these second order systems, the expressions for α and ω_0 may be different. However, their values may always be found by comparison of the system homogeneous equation and Equation 21. Therefore, α and ω_n can be found from the coefficients of the differential equation of any second order system written in this form. The roots of the characteristic equation in terms of α and ω_n are:

$$s_1,\ s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \tag{23}$$

The solution for the natural response then is written:

$$v_n(t) = A_1\, e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_n^{\,2}}\right)t} + A_2\, e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_n^{\,2}}\right)t} \tag{24}$$

Constants A 1 and A2 must be determined by initial conditions of the system.

Note that $\sqrt{\alpha^2 - \omega_n^{\,2}}$ can be real, zero, or imaginary, depending on the relative size of α and ω_n. This leads to three different possible forms for the natural response function, $v_n(t)$. Before we develop these three forms, it will be useful to define an additional useful constant, ζ, the **damping ratio:**

$$\zeta \equiv \frac{\alpha}{\omega_n}.$$

In terms of this parameter, the homogeneous equation is:

$$\frac{d^2 v_n}{dt^2} + 2\,\zeta\,\omega_n \frac{dv_n}{dt} + \omega_n^2\, v_n = 0 \tag{26}$$

In terms of ω_n and ζ, the roots of the characteristic equation become

$$s_1,\ s_2 = \omega_n\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right] \tag{27}$$

If $\alpha > \omega_n$, meaning $\zeta > 1$, the system is **overdamped** and the roots s_1 and s_2 are real, negative, and distinct. (This was the case for the system solved in the earlier numerical example.) An overdamped system responds slowly but smoothly to changes of the input. The natural response takes the form:

$$v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{28}$$

where s1 and s2 are negative, real numbers.

If $\alpha < \omega_n$, meaning $\zeta < 1$, the system is **underdamped**, with complex conjugate roots.

$$s_1,\ s_2 = -\alpha \pm j\sqrt{\omega_n^2 - \alpha^2}$$

$$\text{or} \quad s_1,\ s_2 = -\zeta\,\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \tag{29}$$

An underdamped system in general will respond more quickly to input changes, but may overshoot the final value with oscillation. The natural response in this case will be of the foam shown in Equation 20. (This form will be derived in a later example.)

$$v_n(t) = A\, e^{-\alpha t} \cos(\omega_d t + \theta) \tag{30}$$

$$\text{where} \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\omega_n^2 - \alpha^2} \tag{31}$$

Note that in Equation 20 the real part of the root, α, affects the rate of decay of the function, and the imaginary part of the root, ω_d, determines the oscillatory frequency. Equation 21 indicates that ω_d, the frequency of oscillation with damping present, is less than ω_n ω_n is called the undamped natural frequency because it is the frequency of oscillation if α is zero. Constants A and θ are again determined by initial conditions.

If $\alpha = \omega_n$, then $\zeta = 1$, and the system is **critically damped**. In this case the roots are real, negative, and repeated (equal).

$$s_1,\ s_2 = -\alpha \tag{32}$$

A critically damped system provides the quickest possible response without oscillation. (Note that this case is only a single mathematical point. Any real system is unlikely to be perfectly set to critical damping although this may be a design goal') The natural response for the case of critical damping will be of the form:

$$v_n(t) = B_1 e^{-\alpha t} + B_2 t e^{-\alpha t} \tag{33}$$

where B 1 and B2 are again determined by initial conditions.

Typical plots of over, under, and critically damped second order responses are given in Figure 10. For these plots, ω_n was held constant and ζ was varie. For the overdamped case (ζ = 2), note that the transition from the initial value (12) to the final value (0), is slow and smooth. For the critically damped case (ζ = 1), the transition takes place more quickly, but still there is no oscillation. For the underdamped case (ζ = 1/8), the initial transition is fastest, but it overshoots and oscillates about the final value. Often in design, it is desireable to tolerate some overshoot in order to obtain faster response. A typical choice for ζ might be around 0.7.

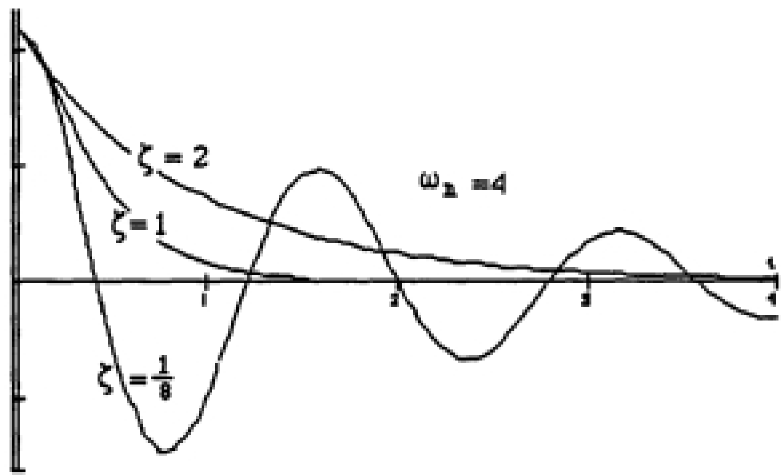


Figure 10 Typical over, under, and critically damped waveforms

We have seen that the three possible forms of the natural response of second order systems (Equations 28, 30, and 33) are related to the type of roots obtained from the characteristic equation. It is informative to plot the location of these roots in the complex plane as the damping of the system is varied. Suppose we hold ω_n fixed, but vary ζ from 0 to ∞. (In the electric system example, this could be done by holding L and C fixed and varying R.) Using Equations 23, 27, or 29, the result is given in Figure 11:

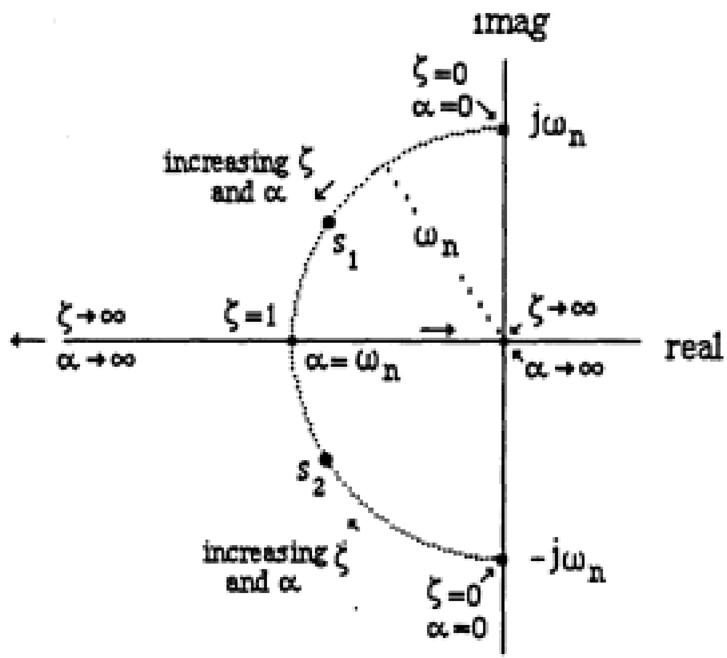


Figure 11 Locus of roots as ζ, α vary from 0 to ∞

When $\zeta < 1$, the roots are complex conjugates. This corresponds to the underdamped condition. Note that the magnitude of either complex root (Equation 29) is a constant equal to ω_n. Therefore, for $\zeta < 1$, the loci form a semicircle intersecting the imag axis at $\pm j\omega_n$ (when $\zeta = 0$), and approaching the real axis at $-\omega_n$ as $\zeta \to 1$.

When $\zeta = 1$, both roots fall on the point $-\omega_n$ ($= -\alpha$), corresponding to the critically damped condition. The characteristic equation now forms a perfect square and its roots are repeated.and equal to $-\omega_n$

As ζ grows larger than 1, one root migrates toward the origin, while the other moves out on the negative real axis toward $-\infty$. This range corresponds to the overdamped condition with two simple exponentially decaying terms in the natural solution.

Example 1
Overdamped System:

In the system below, L = 5H, R = 8Ω, and C = 1/80 F. The capacitor is charged to 3 volts when at t = 0, the switch closes. Find $v_n(t)$ for $t > 0$. (Since there is no forcing function driving the system, only the natural voltage response is possible at any node.

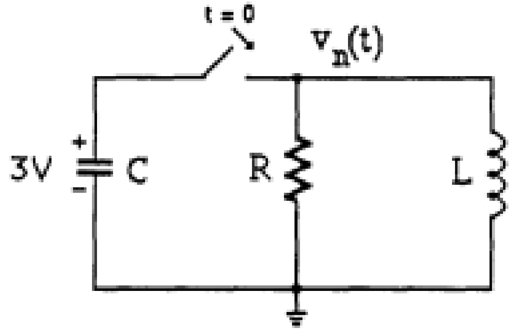


Figure 12 System for damping examples

Solution:

The nodal equation for the top node is:

$$0 = C\frac{dv_n}{dt} + \frac{1}{R}v_n + \frac{1}{L}\int v_n\,dt \tag{34}$$

Substitute the element values and rearrange to get:

$$\frac{d^2v_n}{dt^2} + 10\frac{dv_n}{dt} + 16\,v_n = 0 \tag{35}$$

We have defined the coefficients of a second order equation in terms of damping coefficient and undamped natural frequency (Equation 22):

$$\frac{d^2v_n}{dt^2} + 2\alpha\frac{dv_n}{dt} + \omega_n^2\,v_n = 0 \tag{36}$$

Comparison of Equations 35 and 36, and using Equation 25 yields

$$\alpha = 5 \qquad \omega_n = 4 \qquad \zeta = 5/4 : \quad \text{overdamped} \tag{37}$$

(We recognize that the system is overdamped because $\zeta > 1$.)

The characteristic equation is $s^2 + 10\,s + 16 = 0$. The roots of this equation are $s = -2$ and $s = -8$. Therefore, the natural solution is

$$v_n(t) = A_1\,e^{-2t} + A_2\,e^{-3t}. \tag{38}$$

To find A1 and A2, consider the initial conditions. We were told that the capacitor was charged to 3 volts before the switch was closed. Based on our knowledge of A-type energy storage elements, and the fact that no impulse flows will take place through the resistor or the inductor, we can assume that the capacitor's voltage at t = 0+ is the same as it was at t = 0- (3 volts). Since the capacitor's voltage and vn(t) are equal for t > 0, we have

$$v_c(0^+) = v_n(0^+) = A_1 + A_2 = 3 \tag{39}$$

We need another equation to solve for the two unknown constants, A1 and A2. The second equation can be obtained by using what we know about changes in the other energy storage element, the inductor. As a T-type element, the inductor's current will not change instantaneously unless an impulse voltage is applied across it. Therefore, in this circuit, the inductor's current at t = 0+ must be the same as it was at t = 0-, i.e., zero amps. Now, this is not exactly what we need, since our equation is in terms of the voltage across the system, not the inductor current. However, we can use the inductor's current at t = 0+ to determine the derivative of voltage' We have determined that at t = 0+, the capacitor voltage is 3. This means that the resistor's current must be 3/8 amp downward at t = 0+. Since the inductor's current is zero, the 3/8 amp must *flow* back up through the capacitor' Current in a capacitor determines the derivative of capacitor voltage which, in this case, is the same as the derivative of vn.

$$\frac{dv_n}{dt} = \frac{i_c}{C} \tag{40}$$

Because the charge *flow is* off the capacitor, the derivative is negative, equal to -(3/8) (80) =-30 volts/sec.

A second way to get dv/dt is simply to plug what we know back into Equation 34, evaluated at t = 0+.

$$\frac{1}{80}\,\frac{dv_n}{dt}(0^+) + \frac{1}{8}\,v_n\,(0^+) + \frac{1}{5}\int_{0^-}^{0^+} v_n\,(t)\,dt = 0 \tag{41}$$

The integral term is zero because the limits are infinitessimally close and vn(t) is not an impulse function. The second term is 3/8 because we already have found that vn(0+) = 3. Then, solving Equation 41 for dvn/dt at t = 0+ yields the same - 30 volts/sec found above. Now, differentiate Equation 38, evaluate at t = 0+, and set equal to -30:

$$-2\ A1 - 8\ A2 = -30 \tag{42}$$

Solve Equations 39 and 42 simultaneously to get A 1 = -1 and A2 = 4. The final result for the natural voltage appearing across the circuit is:

$$v_n(t) = [\ -e^{-2t} + 4\,e^{-3t}.]\ ,\ t > 0 \tag{43}$$

The t > 0 designation is needed because the system equation we have solved (Equation 34) does not apply with the switch open. If the voltage across R or L were being expressed, a unit step function could have been used in place of t > 0, since we know these elements had zero voltage prior to t = 0. A plot of the two terms of Equation 43 as well as their sum is given in Figure 13' Note that the two components of vn have different time constants, corresponding to the critical frequencies of the system. The contribution of the 1/8 time constant function is

over by t = .75. After this, v_n is very close to $- e^{-2t}$.

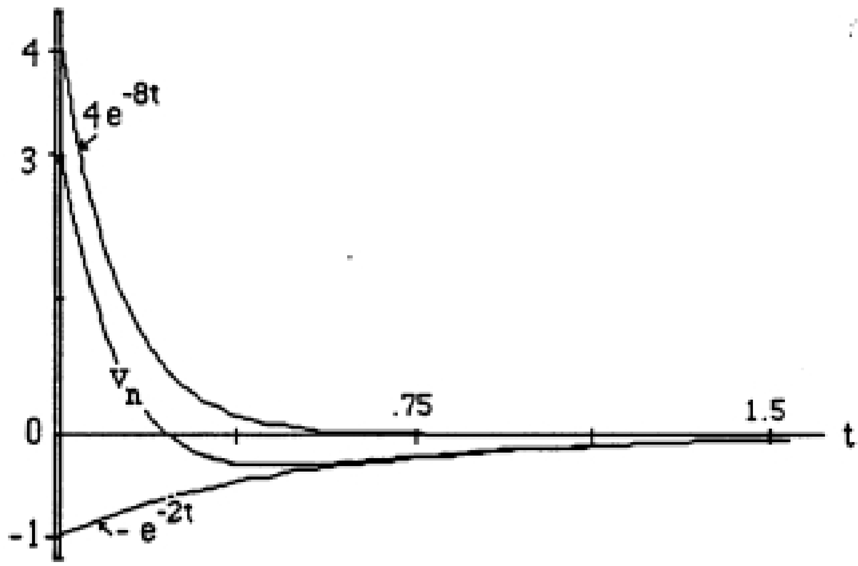


Figure 13 Components of natural response of Example 1

It is important to note here that all variables in the system, not just the voltage, have the same form as Equation 38. Only the constants A 1 and A2 may be different if the response is defined as the resistor current or the inductor current, etc. In a given linear system with specified forcing function locations, the A and T variables of all elements are identical except for the value of the constants of integration.

Example 2
Critical Damping

We now repeat the same problem as Example 1, but with element values changed to C = 0.5F, L = 1H, and R = $1/\sqrt{2}\Omega$.

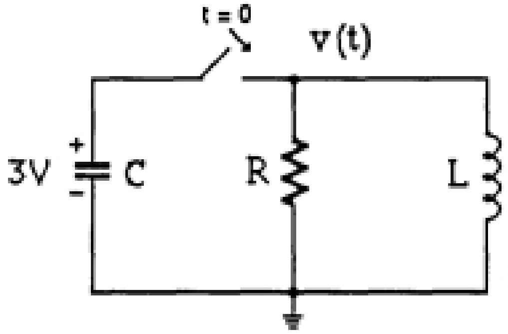


Figure 14 System for Example 2

We write the same nodal equation (Equation 34) at the top node' When element values are substituted, the equation now becomes

$$\frac{d^2 v_n}{dt^2} + 2\sqrt{2}\,\frac{dv_n}{dt} + 2\,v_n = 0 \tag{44}$$

From which the characteristic equation is

$$s^2 + 2\sqrt{2}\,s + 2 = 0 \tag{45}$$

The expression on the left is a perfect square. The roots of Equation 45 are therefore repeated, both equal to the negative square root of 2. The problem with this result is that we can't use the general form of Equation 7 because that would lead us to Equation 46 which allows only one independent constant of integration. (We know that for a second order system, there must be two')

$$A_1 e^{-\sqrt{2}\,t} + A_2\, e^{-\sqrt{2}\,t} = A_3\, e^{-\sqrt{2}\,t} \tag{46}$$

The problem arises because the roots are exactly equal. We know that if the roots are slightly different, two distinct terms will result. To find what happens as the roots approach equality, we will assume a slight difference in the roots and then examine how the solution changes as the roots come closer. Suppose one root is S1 and the other is s2 $+ \varepsilon$. Then from Equation 17 we have

$$v_n = A_1 e^{s_1 t} + A_2\, e^{(s_1 + \varepsilon)t} \tag{47}$$

The expansion of e $^{\varepsilon t}$ into an infinite series is given in Equation 48:

$$e^{\varepsilon t} = 1 + \varepsilon t + \frac{\varepsilon^2 t^2}{2!} + \cdots \tag{48}$$

Substituting this series for $e^{\varepsilon t}$ of Equation 47 yields

$$v_n = (A_1 + A_2)\, e^{s_1 t} + A_2 e^{s_1 t} \left(\varepsilon t + \frac{\varepsilon^2 t^2}{2!} + \cdots \right) \tag{49}$$

Now define $B_1 = (A_1 + A_2)$ and $B_2 = A_2 \varepsilon$.

$$v_n(t) = B_1 e^{s_1 t} + B_2 e^{s_1 t} \left(t + \frac{\varepsilon t^2}{2!} + \cdots \right) \tag{50}$$

As a approaches zero (meaning the two roots approach equality), the quantity in the parentheses becomes just t. Therefore, the form of the natural solution for a second order system with repeated roots is

$$v_n(t) = B_1 e^{s_1 t} + B_2 t e^{s_1 t} \tag{51}$$

Returning to the RLC circuit problem, we see from Equation 51 that the solution is

$$v_n(t) = B_1 e^{-\sqrt{2}\, t} + B_2\, t\, e^{-\sqrt{2}\, t} \tag{52}$$

The values of constants B 1 and B2 must again be found using initial conditions of the voltage and its derivative at t = 0+. Because the capacitor is charged to 3 volts at t = 0', we know that vn(0+) = 3. Substituting into Equation 24 at t = 0+ gives B 1 = 3.

To find the derivative of the voltage at t = 0+, substitute what we know back into the original node equation (similar to the method usred in Equation 41 for the underdamped case):

$$\frac{1}{2} \frac{dv_n}{dt}(0^+) + \sqrt{2}\,(3) + 0 = 0 \tag{53}$$

From Equation 53 we see that the initial value of the derivative is $-6\sqrt{2}$. Differentiation of Equation 51 yields

$$\frac{dv_n}{dt} = -\sqrt{2}\, A_1 e^{-\sqrt{2}\, t} - \sqrt{2}\, A_2 t\, e^{-\sqrt{2}\, t} + A_2 e^{-\sqrt{2}\, t} \qquad (54)$$

A_1 has been found to be 3. Evaluate Equation 44 at $t = 0^+$ and set it equal to $-6\sqrt{2}$ (found from Equation 53) to find A_2. The result is $A_2 = -3\sqrt{2}$. The solution for the natural response is then

$$v_n(t) = 3\, e^{-\sqrt{2}\, t} - 3\sqrt{2}\, t\, e^{-\sqrt{2}\, t} \quad , t > 0. \qquad (55)$$

This function is plotted in Figure 15.

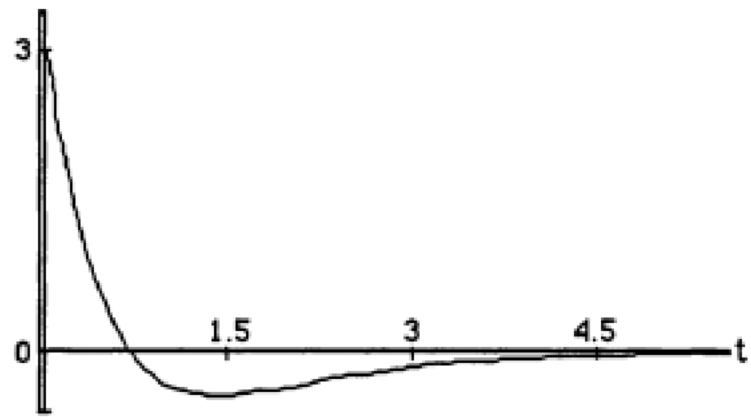


Figure 15 Critically damped vn(t) from Example 2

Example 3
Underdamped Case

We repeat the same example with new element values. A capacitor, charged to 3 volts is connected to a parallel RLC circuit at $t = 0$. This time the element values are $R = 169\ \Omega$, $L = 10$ H, and $C = 1/1690$ F. The circuit is again presented in Figure 16.

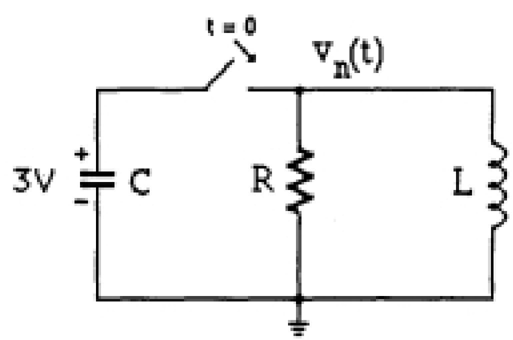


Figure 16 System for Example 3

The differential equation for $v_n(t)$ is again

$$0 = C\frac{dv_n}{dt} + \frac{1}{R}v_n + \frac{1}{L}\int v_n\,dt \tag{56}$$

Differentiate and divide by C:

$$\frac{d^2v_n}{dt^2} + 10\frac{dv_n}{dt} + 169 = 0 \tag{57}$$

The values of α, ζ, ω_n, and ω_d can be obtained by finding the roots of the characteristic equation, $s^2 + 10s + 169 = 0$, which are $-\alpha \pm j\,\omega_d$, and from the coefficients of Equation 57, along with Equations 22 and/or 26. From Equation 26 we see that $\omega_n = 13$. The roots of the characteristic equation are $-5 \pm j12$. Therefore, $\alpha = 5$, $\omega_d = 12$, and $\zeta = 5/13$. Before proceeding with the numerical solution, we will investigate the effect of comple conjugate roots:

For complex conjugate roots, we have defined α and ω_d such that

$$s_1, s_2 = -\alpha \pm j\,\omega_d \tag{58}$$

The solution for the natural response is then

$$v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-\alpha t + j\omega_d t} + A_2 e^{-\alpha t - j\omega_d t} \tag{59}$$

Factor out $e^{-\alpha t}$:

$$v_n(t) = e^{-\alpha t}\left(A_1^{+j\omega_d t} + A_2 e^{-j\omega_d t} \right) \tag{60}$$

Now use Euler's Identity to covert to sines and cosines:

$$v_n(t) = e^{-\alpha t}\left(A_1 \cos\omega_d t + j A_1 \sin\omega_d t + A_2 \cos\omega_d t - j A_2 \sin\omega_d t \right)$$

$$v_n(t) = e^{-\alpha t}\Big((A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\Big) \tag{61}$$

Define new constants $B_1 = A_1 + A_2$, and $B_2 = j\,(A_1 - A_2)$:

$$v_n(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right) \tag{62}$$

Equation 62 can also be written in the form of Equation 20:

$$v_n(t) = K e^{-\alpha t} \cos(\omega_d t + \theta) \tag{63}$$

$$\text{where } K = \sqrt{B_1^2 + B_2^2} \quad \text{and} \quad \theta = -\tan^{-1}\frac{B_2}{B_1}$$

Now returning to Example 3, to solve for $v_n(t)$, we can use either Equation 62 or 63. In either case, we must evaluate two constants (B_1 and B_2 or K and θ). Usually, the easier one to use is Equation 62. Again we use initial conditions to find these constants. Because the capacitor is an A-type element, its voltage at $t = 0^+$ (and therefore $v_n(0^+)$) will be 3 volts. Equation 62 at t = 0 is $3 = B_1$. For B_2 we need the derivative of $v_n(t)$ at $t = 0^+$. Once again, there are two ways to get this value: from a consideration of the system elements and from the system differential equation.

When the capacitor is connected, its 3 volts causes a 3/169 amp current in the resistor at $t = 0^+$. No current flows in the inductor because of the rule concerning step changes of the T variable in T-type elements (the inductor's current at $t = 0^-$ was 0). So 3/169 must also be the current out of the capacitor. The rate of change of the capacitor's voltage is always i_c/C. So

$$\frac{dv_n}{dt}(0^+) = \frac{i_c(0^+)}{C} = \frac{\frac{-3}{169}}{\frac{1}{1690}} = -30\,\frac{\text{volts}}{\text{sec}} \tag{64}$$

An alternative way to obtain the initial value of the derivative is to evaluate the differential equation (Equation 56) at $t = 0^+$:

$$0 = C\frac{dv_n}{dt} + \frac{1}{R}v_n + \frac{1}{L}\int v_n\,dt \tag{65}$$

$$0 = \frac{1}{1690}\frac{dv_n}{dt}(0^+) + \frac{1}{169}(3) + 0 \tag{66}$$

Equation 66 gives the same value for the derivative at $t = 0^+$, -30 volts/sec.
Differentiate Equation 62 (you must differentiate it as a product!), evaluate it at t = 0, and

set it equal to -30. The value of B_2 is then found to be -15/12. The solution for vn(t) is then:

$$v_n(t) = e^{-5t}\left(3 \cos 12\,t \;-\; \frac{15}{12}\; \sin 12\,t\right), \quad t > 0 \tag{67}$$

This can also be expressed in the form of Equation 30,

$$v_n(t) = 3.21\, e^{-5t} \cos\,(12\,t - 21^{o}) \tag{68}$$

<u>complete response for second order systems</u>

In the previous examples, the systems studied had no forcing function' Therefore, only the natural solution resulted. When a forcing function is included, the forced response must be added to the natural response before the constants are evaluated. The evaluation of the two constants is done in the same manner as in the previous examples, using initial conditions. The form of the forced response matches the forcing function. That is, a sinusoidal forcing function produces a sinusoidal forced response, a pulsed forcing function produces a pulsed forced response, etc. For now, we will consider only constant forcing functions. Later, we will study methods by which any forcing function may be handled.

Consider Equation 69, a general equation for a second order system system. The natural response, if substituted for y on the left side, would by definition cause those three terms to add to zero. The forced response, y_f, is that function which when substituted on the left side, causes the terms to add to x(t) for t > 0.

$$a_2 \frac{d^2y}{dt^2} \;+\; a_1 \frac{dy}{dt} \;+\; a_0 y \;=\; x\,(t) \tag{69}$$

Exact determination of y_f for any general x(t) can be difficult. However, the form of the forced response to a constant driving force can be easily found from the system equation or from our knowledge of the behavior of the system elements.

Suppose x(t) is a constant (X0) for t > 0. The terms on the left of Equation 69 must add to X0 for all t > 0. We also expect the forced response yf to be a constant. If a constant if is substituted on the left side of Equation 69, all derivative terms on the left become zero, leaving only the a0yf term equal to X0. Therefore

$$y_f = X_o / a_o \tag{70}$$

If the system equation contains terms which contain integrals of the dependent variable, as in Equation 71, the forced response must be zero. This must be the case because any non-zero constant under the integral sign would cause that term to grow without limit'

$$f(t) = C\,\frac{dv}{dt} \;+\; \frac{1}{R}\; v \;+\; \frac{1}{L}\int v\,dt \tag{71}$$

For example, in the circuit of Figure 17, suppose f(t) is a unit step function, u(t).

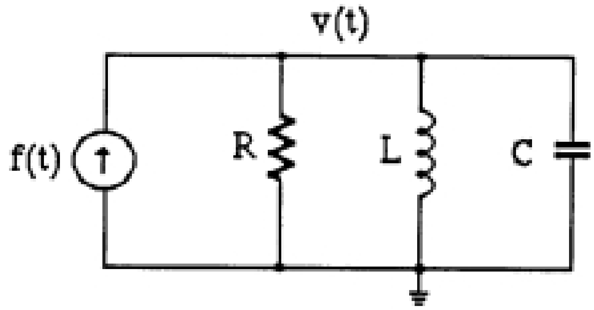


Figure 17 Second order system driven by a constant f source.

Equation 71 is a nodal equation written at the top node of Figure 17' Because f(t) is a constant, we expect vf(t) also to be a constant for t > 0. 11-1 addition, vf(t) must solve Equation 71 for all t>0. But if vf(t) is a non-zero constant, the integral term will grow without limit' Therefore, vf(t) = 0 for this case.

Example

As a final example, find the complete response of the system of Equation 72. We are given that y(0+) =1 and that di/dt at t = 0+ = 2.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 5\,u(t) \tag{72}$$

Solution:

1. Find the form of the natural response:

s2 + 4s + 5 = 0 has roots s l, s2 = - 2 ± j' Therefore, the system is underdamped. We have found that the natural solution can be written in either of two forms:

$$y_n = A_1 e^{(-2-j)t} + A_2 e^{(-2+j)t} = K e^{-2t}\cos(t+\phi) \tag{73}$$

Although the second expression of Equation 73 may look nicer, using the initial conditions to solve for the constants K and 0 is generally more difficult than finding A1 and A2 of the first expression.

2. Find the forced response:

Because the forcing function is a constant, we can assume the forced response is a constant. Then all terms on the left side of Equation 72 are zero for t > 0, except 5y which must equal the other side, 5 for t > 0. So the forced response is

yf = 1, t>0.

3. Determine the constants of integration.

The complete response is the sum of the natural and forced responses.

$$y(t) = y_n + y_f = A_1 e^{(-2-j)t} + A_2 e^{(-2+j)t} + 1, \quad t > 0 \tag{73}$$

The constants A_1 and A_2 are determined from the initial conditions, just as was the case in the previous examples of non-driven systems. The difference is that the forced response must be included in the evaluation of these constants.

The initial values of y and dy/dt were given. Use Equation 73 to obtain the following equations which can be solved for A_1 and A_2.

$$y(0) -1=A1+A2+1$$

$$\frac{dy}{dt}(0) = 2 = (-2-j)A_1 + (-2+j)A_2 \tag{74}$$

Simultaneous solution of Equations 74 yields $A_1 = j$ and $A_2 = -j$. Substitution of these results back into Equation 73 gives the solution for y(t):

$$y(t) = j e^{(-2-j)t} - j e^{(-2+j)t} + 1$$

$$= e^{-2t}\left[j e^{-jt} - j e^{+jt}\right] + 1$$

$$= e^{-2t}\left[\frac{e^{jt} - e^{-jt}}{j}\right] + 1$$

$$y(t) = 1 + 2 e^{-2t} \sin t, \quad t > 0.$$

Chapter 9 Problems

1. For each second order equation below,

(a) Sketch the location of roots of the homogeneous equation on the

(b) Determine the values of α, ω_n and ζ.

(c) State whether the system is over, under, or critically damped. If underdamped, calculate wd.

(d) For each case, sketch the general shape of the unit step response of the system. (It should not be necessary to completely solve the equation analytically.)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u(t)$$

$$2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 28y = u(t)$$

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = u(t)$$

2. Below are input-output equations written for four second order systems. For each equation:

a. Write an expression for the natural response of the system. The expression must include numerical values for all frequencies (roots of s).

b. Estimate the time it will take for the natural response to die out.

c. Determine ζ and ω_n.

d. Specify under-, over-, or critically-damped.

e. If $x(t) = u(t)$ and systems were initially "dead", determine the complete response.

f. Check your results with Maple.

$$2\frac{d^2y}{dy^2} + 98y = x(t) \qquad \frac{d^2y}{dy^2} + 11\frac{dy}{dt} + 18y = x(t)$$

$$4\frac{d^2y}{dy^2} + 40\frac{dy}{dt} + 100y = x(t) \qquad 2\frac{d^2y}{dy^2} + 8\frac{dy}{dt} + 16y = x(t)$$

3. The suspension system of a car may be represented as shown in the figure below.

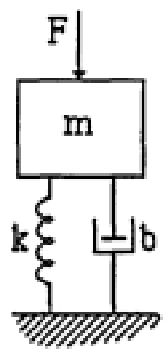
M represents the mass of the car, and k and b are, respectively, the effective spring and damping constants of the suspension system.

$M = 1000$ kg, $k = 25{,}000$ N/m, and $b = 6{,}000$ Ns/m.

(a) If you push down suddenly on this car's bumper (force F), will it tend to (1) move slowly to a new equilibrium position? or (2) oscillate up and down for a while?

(b) If (1), approximately how long will it take to reach its new equilibrium? If (2), with what radian frequency will it oscillate?

(c) What value of damping should be used with this mass and spring so that the system makes the fastest return to equilibrium without oscillation?



Systems Chapter 10 Study Guide

Impulse and Step Response

A. Concepts Addressed By This Chapter

1. Unit impulse response of a system.
2. Unit step response of a system

B. Introduction

The unit impulse response of a system is defined as the response it exhibits when a unit impulse forcing function is applied at t = 0 to the system in which all elements have zero stored energy. While an impulse may not seem to be a very realistic driving force, we will see that most real functions can be approximated by combinations of impulses. Furthermore, the waveform of a system's impulse response contains sufficient information to permit the determination of the system's response to any input waveform.

The unit step response has the same definition as the impulse response except for replacement of the impulse driver by a step function driver. The step response is equally useful as a source of information about the system's behavior to other inputs.

The impulse response is the derivative of the step response.

C. Instructional Objectives

A student mastering this material will be able to

1. Given a system diagram or a differential equation, determine the unit impulse response.
2. Given a system diagram or a differential equation, determine the unit step response.
3. Determine the initial (t = 0+) values of all variables.

D. Study Procedure

Read Chapter 10 of these notes.

Further information on impulse and step response can be found in references 1, 6, 13.

E4 Systems
Chapter 10
Impulse and Step Response

1. Impulse Response

The "impulse response" of a one-input, one-output, linear, time-invariant system is defined as the output function which appears under conditions of zero initial energy storage (zero state response), when a unit impulse forcing function is applied at the input at t = 0.

Consider a general system where the equation relating output y(t) and input x(t) is:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = x(t) \tag{1}$$

If x(t) is an impulse, this is a homogeneous equation for t > 0 because the impulse is zero for t > 0. Therefore, the response to an impulse is the natural response, yn. To find the impulse response, we need to find the natural response, then evaluate the constants of integration (one for a first-order system, two for a second-order system, etc., to obtain the magnitude of the response). As an example, suppose we have a second order system with an impulse input:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = \delta(t) \tag{2}$$

At t = 0, the right side is an impulse. Therefore one or more terms on the left must be an impulse at t = 0. If y is $\delta(t)$, dy/dt and d^2y/dt^2 will be $\delta'(t)$ and $\delta''(t)$, respectively. Since no terms like these appear on the right side, we can assume y is not an impulse, nor is dy/dt. Therefore only the highest derivative term can be an impulse at t = 0. Since it is the only impulsive term, $a_2(d^2y/dt^2) = \delta(t)$. (Both dy/dt and y are finite and therefore negligible at t=0 when compared to the impulsive d^2y/dt^2.) We can summarize this reasoning:

If the forcing function is an impulse, the highest derivative term must equal the impulse at t=0. The other terms must be finite at t = 0.

For t > 0, the equation is homogeneous (the impulse is over at t=0+ so the right side of Equation 2 is zero). To obtain the solution of this second order equation, we need the values of dy/dt and y at t = 0+. The value of both of these quantities at t = 0- (just before the impulse) is zero since impulse response, by definition, means the system has no energy storage at t = 0-. Values of dy/dt and y at t = 01, can be found by integrating the equation across t = 0 (from 0' to 0+). This process is best illustrated by an example:

Example: Find the impulse response of a system whose equation is:

$$2 \frac{d^2 y}{dt^2} + 10 \frac{dy}{dt} + 12\, y = x\,(t) \tag{3}$$

Since the impulse response has the form of the natural response, find

y_n as we have done previously: The characteristic equation $2s^2 + 10s + 12 = 0$ has roots -3 and -2. Therefore, the impulse response is

$$h(t) = y_n(t) = A_1 e^{-2t} + A_2 e^{-3t}, \quad t > 0.$$

The values for A_1 and A_2 must be determined from initial ($t=0^+$) conditions on y and dy/dt. We know y and dy/dt are zero at $t = 0^-$ (definition of impulse response). Therefore, integrate Equation 3 from a known state ($t = 0^-$) to the desired state ($t = 0^+$).

$$2\int_{0^-}^{0^+} \frac{d^2y}{dt^2}\,dt + 10\int_{0^-}^{0^+} \frac{dy}{dt}\,dt + 12\int_{0^-}^{0^+} y\,dt = \int_{0^-}^{0^+} \delta(t)\,dt \tag{4}$$

The integral of the second derivative (first term on the left side of Equation 4) is the first derivative' Since dy/dt and y are finite at t = 0, the other terms on the left side are zero. On the right side, the area under a unit impulse function is 1. Integrating and substitution of the limits gives:

$$\left[2\frac{dy}{dt}(0^+) - 2\frac{dy}{dt}(0^-)\right] + 0 + 0 = 1 \tag{5}$$

Since dy/dt = 0 at t = 0', we can solve for dy/dt at t = 0+.

$$\frac{dy}{dt}(0^+) = \frac{1}{2} \tag{6}$$

So dy/dt jumps from zero to 1/2 at t = 0'

Since y(0') = 0, and y(0+) is the integral of dy/dt (a step) from t = 0' to 0+, then y(0+)

To find A1 and A2, use the t = 0+ values just determined, together with the general homogeneous solution:

$$h(t) = y(t) = A_1 e^{-2t} + A_2 e^{-3t}$$

$$h(0^+) = y(0^+) = 0 = A_1 + A_2.$$

$$\frac{dh}{dt}(0^+) = \frac{1}{2} = -2A_1 - 3A_2$$

Solving Equations 8 and 9 yields AI = 1/2 , A2 = -1/2. Therefore, the impulse resp(

$$h(t) = (1/2\, e^{-2t} - 1/2\, e^{-3t})\, u(t) \tag{10}$$

In many cases, the input need not be a true impulse to produce an output which approximates an impulse response. The impulse response is an accurate representation of the system's output for any pulse input of unit area whose duration is short compared to time constants of the sysytem. For example, consider the system shown in Figure 1 in which R = 50 Ω and C = 1 µF. It is a first order system with a time constant, $\tau = RC = 50$ seconds. The step response of this system is $(1 - e^{-0.02t})\, u(t)$.

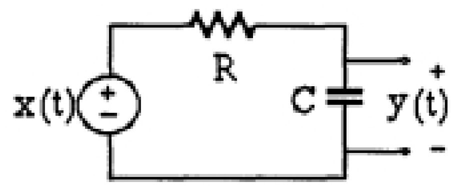


Figure 1 Impulse applied to an RC system

Suppose x(t) is a rectangular pulse of amplitude 0.2, and width 5 seconds (area = 1). During the 5 seconds of the pulse, the system would be exhibiting its step response, the output climbing toward a final value of 0.2 volt. (Of course, it would take 200 seconds or so to get near 0.2 volt.) After 5 seconds, the output has only risen to $0.2(1 - e^{-1/10}) = .0190325$ volt. At this time the input pulse ends, and the voltage source becomes zero (a short circuit). Of course the capacitor voltage does not instantaneously change, but remains .0190325 volt (between $t = 5^-$ and at $t = 5^+$). For t > 5, the capacitor will discharge back through the resistor and its voltage will decay from .0190325 with a time constant RC = 50.

Suppose x(t) had been a pure impulse instead of the rectangular 5-second pulse we just investigated. The impulse response of this system is $h(t) = 1/50\, e^{-t/50}\, u(t)$. At t=5 this is .0181 and is decaying with a time constant of 50. The two responses are plotted in Figure 2:

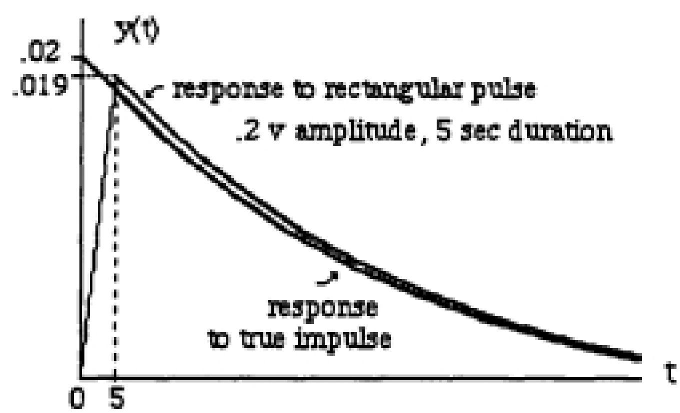


Figure 2 Impulse response compared to the system's response to a short rectangular pulse of unit area. They are similar after the pulse ends.

The two responses are very close for t > 5. If the width of the rectangular input pulse is reduced while its area is held constant, the output will come still closer to a true impulse response. The exact shape of the x(t) input pulse is not important. Any shape which ends in a time much shorter than the system time constant and whose area is unity will produce a response which approximates h(t). This means that in practice, a true impulse input is not required to produce the system's impulse response for analysis. For example, assuming the pulses shown in Figure 3 all have unit area, and all end by t = 5, they will all produce good approximations of the impulse response for t > 5 if they are input to the above system, or to any system whose time constant is long compared to 5

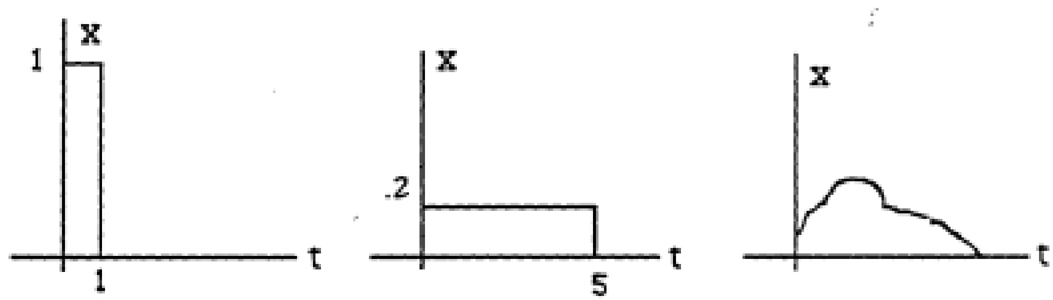
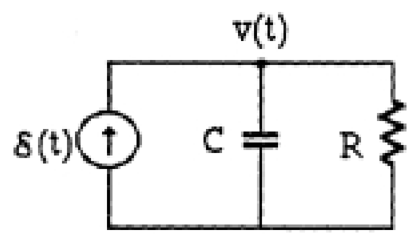


Figure 3 Any pulse of unit area and of duration short compared to the system time constant will produce a response which approximates the impulse response.

Example 1: T-type impulse source applied to an RC parallel system'



The differential nodal equation written at the top node

$$\delta(t) = C\frac{dv}{dt} + \frac{v}{R}$$

The characteristic equation is

$$Cs + \frac{1}{R} = 0 \quad \text{or} \quad s = -\frac{1}{RC}$$

So the natural (zero state) solution is

$$h(t) = v_n(t) = A\,e^{-\frac{1}{RC}t}\,u(t)$$

We need h(t) at t = 0+. We know that at t = 0, the highest derivative term matches the impulse. We integrate each term of Equation 11 from t = 0' to t = 01' Since the v term is not an impulse at t = 0, the integral of this term is zero. Then,

$$C\frac{dv}{dt}(0^+) - C\frac{dv}{dt}(0^-) + 0 = 1 \tag{14}$$

Now dv/dt is zero for t≤0 ⁻, so dv/dt at t = 0+ must be equal to 1/C.. Since C dv/dt is the capacitor's current, apparently all the impulse current from the source takes the capacitor path at t = 0. The charge moved by this current is the area under the current which is 1 for a unit impulse. One coulomb deposited on the capacitor's plates at t = 0 causes the capacitor voltage to jump from 0 volts at t = 0' to 1/C volts at t = 0+.

$$v_n(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i_C dt = \frac{1}{C}\int_{0^-}^{0^+} \delta(t)\, dt = \frac{1}{C} \tag{15}$$

The voltage of Equation 15 is called vn because for t > 0, the input is zero and so any output during this period must be the natural response. So A of Equation 13 is 1/C.

Therefore,

$$h(t) = \frac{1}{C}e^{-\frac{1}{RC}t}\ u(t) \tag{16}$$

Example 2:

An impulse velocity input is applied to the left side of the damping element of Figure 5. Find h(t), the impulse response, which in this case is defined to be the velocity of the mass, vm(t).

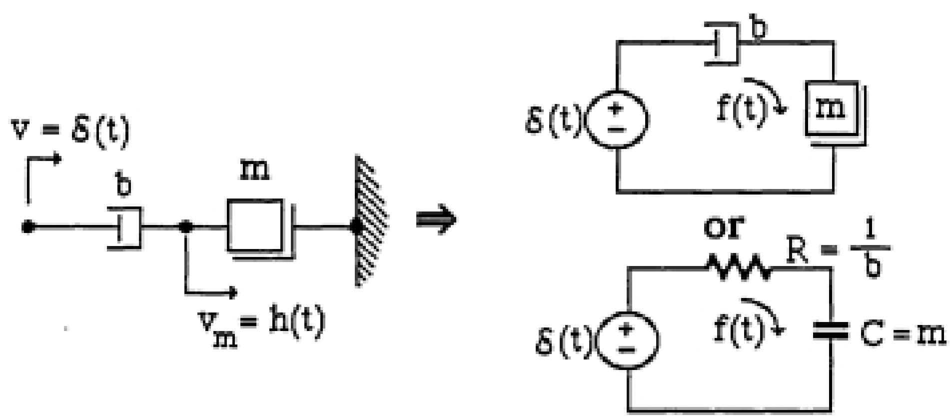


Figure 5 Representations of the system of Example 2

The path (loop) equation is

$$\delta(t) = \frac{1}{b} f(t) + \frac{1}{m}\int_{0^-}^{t} f(t)\, dt + v_m(0^-) \tag{17}$$

The last term is zero because i(0) must be finite. The highest derivative term matches the impulse:

$$f(t) = b\ \delta(t) \ \text{ at } t = 0. \tag{18}$$

The homogeneous Equation is

$$0 = \frac{1}{b} f(t) + \frac{1}{m}\int_{0^-}^{t} f(t)\, dt$$

Differentiating,

$$\frac{df}{dt} + \frac{b}{m} f = 0 \tag{19}$$

The characteristic equation is s + b/m = 0, and the homogeneous solution is

$$f(t) = A_1\, e^{-\frac{b}{m}t}, \ t > 0. \tag{20}$$

This is a solution for f(t). We want vm(t). But in a linear system, all variables follow the same function, only differing in coefficients. So from Equation 20 we can use a new constant and write:

$$v_m(t) = A_2\, e^{-\frac{b}{m}t}, \ t > 0. \tag{21}$$

We now need the value of A2 and the nature of vm(t) at t = 0 (in case it is an impulse). We can find vm(0+) from the expression

$$v_m(0^+) = \frac{1}{m}\int_{0^-}^{0^+} f(t)\, dt \tag{22}$$

This is simply the basic force-velocity relation for a mass. In this case, the force was zero for t < 0, so the integral can start at t = 0-. From Equation 18 we see that f(t) is an impulse occurring within the limits of this integral. Therefore, From Equations 18 and 22, we see that vm(t) is not an impulse at t = 0' (This is also evident from the rule we developed about infinite energy storage.) Therefore, the impulse response is

$$vm(t) = h(t) = b/m\, e^{-b/m\, t}\ u(t) \tag{24}$$

Here, the step function is used since we know all velocities were zero for t < 0.

Example 3 Impulse response of a second order system

' Find the impulse response of the RLC circuit shown if the response is defined to be the voltage across the circuit and the impulse is a current source driver.

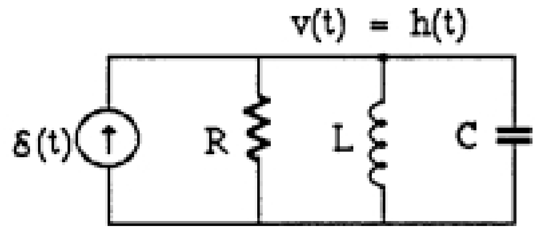


Figure 6 RLC parallel circuit with an impulse current applied

The vertex law equation at the top node is:

$$\delta(t) = C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int_{0^-}^{t} v\,dt \tag{25}$$

Note that the three terms on the right of Equation 25 represent current in the capacitor, current in the resistor, and current in the inductor, respectively. There are two ways to proceed at this point in most impulse response problems. One way, as was used in Examples 1 and 2, is to use the system equation to obtain the required initial conditions. A second approach is to use our Knowledge of the behavior of R, L, and C ideal elements. This method will be illustrated here. The inpulse current arriving at the top node must flow down through R, L, and/or C. Which path or paths does it take? Assume the impulse current (or part of it) flows through L. Then the voltage across L, vL = L di/dt, is a doublet' This voltage is also across the other elements, producing a doublet current in R and the derivative of a doublet current in C. Since these currents do not add to zero at the top node, the assumption of an impulse current in L cannot be correct. Does the impulse current from the source flow down through R? This would produce an impulse voltage across R which would also be across C. An impulse voltage across C produces a doublet current in C. But there is no doublet current in any other element to satisfy the current law at the top node. Therefore, the only possible path for the impulse current delivered by the source in Figure 6 is down through the capacitor'

If a unit impulse current flows in a capacitor, the capacitor's voltage jumps by 1/C times -the area under the impulse (the charge)' In the case of this example, since the capacitor's voltage is zero when the impulse arrives, the voltage steps from zero volts at t = 0_, to 1/C volts at t = 0+'

$$v_C(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i\,dt = \frac{1}{C}\int_{0^-}^{0^+} \delta(t)\,dt = \frac{1}{C}\text{ volts} \tag{26}$$

A step voltage across all elements at t = 0 produced by the impulse current's taking the capacitor path does not result in any violations of path or vertex rules as did the previous assumptions that the impulse took an R or L path. A rule we can glean from this exercise is that An <u>impulse current (or any current) will always take the path which requires the least volatge </u>An impulse current needs a doublet voltage to flow in an inductor, an impulse voltage to flow in a resistor, and only a (finite) step voltage to flow in a capacitor.

We need two initial conditions to solve this second order system' The first is vC(0+) which we have just found to be 1/C volts. What about dv/dt at t = 0+? At this point, the impulse is over and the circuit is as shown in Figure 7.

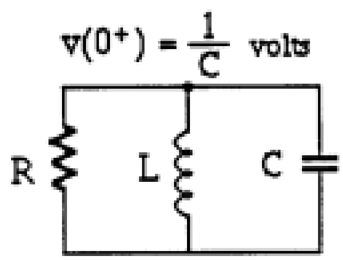


Figure 7 System at t = 0+

Now dv/dt is equal to ic/C from the defining equation of a capacitor. So, if we find iC(0+), we have the initial value of dv/dt' At t = 0+, 1/C volts is across R, causing a resistor current of 1/C amps, downward' The inductor current at t = 0+ is zero because v(0+) is finite and iL was zero at t = 0'. (Remember, inductor current cannot change instantaneously unless there is an impulse voltage appalled.) Therefore, the current flow at the instant t = 0+ is as indicated in Figure 8.

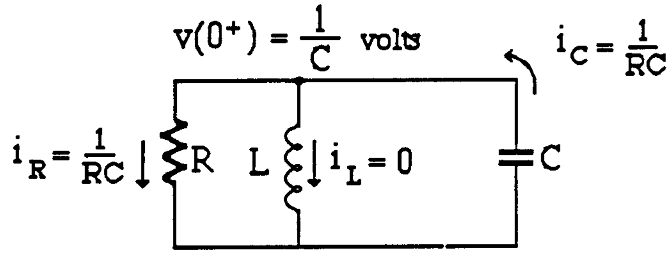


Figure 8 System and currents at the instant t = 0+

Current flowing <u>from</u> the capacitor top plate causes its voltage to decrease (dv_C/dt is negative.

$$\frac{dv}{dt}(0^+) = \frac{i_C(0^+)}{C} = -\frac{1}{RC^2} \tag{27}$$

The two required initial conditions are then

$$v(0^+) = \frac{1}{C} \quad \text{and} \quad \frac{dv}{dt}(0^+) = -\frac{1}{RC^2} \tag{28}$$

The impulse response of a system is simply the solution of its homogeneous equation using initial conditions found as indicated above.

Example 4

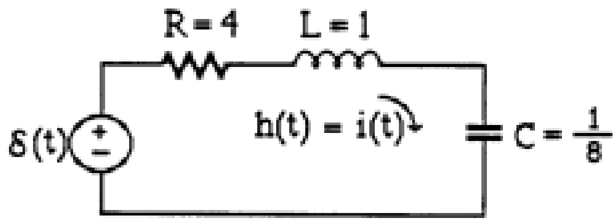


Figure 9 System of Example 4

Find the impulse response of the RLC series system shown' Response is defined to be i(t).

$$\delta(t) = L\frac{di}{dt} + R\,i + \frac{1}{C}\int_{0^-}^{t} i\,dt = \frac{di}{dt} + 4\,i + 8\int_{0^-}^{t} i\,dt \tag{29}$$

$s^2 + 4\,s + 8 = 0$ yields roots $-2 \pm j2$, so the system is underdamped, and the homogeneous solution can be written in either form of Equation 30.

$$i(t) = C\,e^{-2t}\cos(2t + \theta) \quad \text{or} \quad i(t) = e^{-2t}\,(B_1 \cos 2t + B_2 \sin 2t) \tag{30}$$

To find B1 and B2, we need initial conditions.

$L\frac{di}{dt} = \delta(t)$ at $t = 0$. Integrate both sides from $t = 0^-$ to $t = 0^+$:

$$i(0^+) = \int_{0^-}^{0^+} \frac{di}{dt}\,dt = \frac{1}{L} = 1 \text{ amp} \tag{31}$$

Substitute $t = 0^+$ into the second form of Equation 30 and, using the result of Equation 31,

$$B_1 = 1 \tag{32}$$

at $t = 0^+$, Equation 29 becomes $0 = L\frac{di}{dt}(0^+) + R\,(1) + \frac{1}{C}\int_{0^-}^{0^+} i\,dt$ (33)

In Equation 33, the integral is zero because i(0) is finite. Therefore we can solve this equation for di/dt:

$$\frac{di}{dt}(0^+) = -\frac{R}{L} = -4 \tag{34}$$

Differentiating Equation 30 yields

$$\frac{di}{dt}(0^+) = -2\,B_1 + 2\,B_2 = -4 \tag{35}$$

Substitution of the results of Equation 32 gives the value of B_2:

$$B_2 = 1 \tag{36}$$

The system impulse response is given in Equation 37 and plotted in Figure 10.

$$h(t) = 4.123\, e^{-2t} \cos(2t + 76^{o}) \tag{37}$$

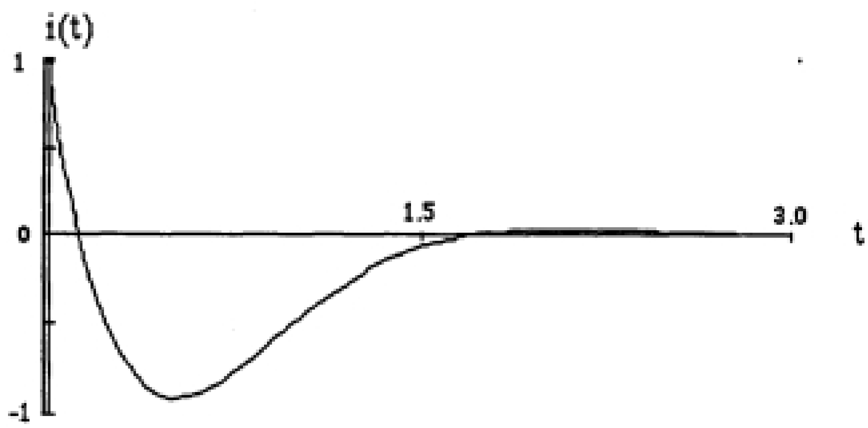


Figure 10 Impulse response of the system of Figure 9.

2. Step Response

The unit step response of a one-input, one-output, linear, time-invariant system is defined in a way similar to the impulse response. The unit step response is the output function which appears under conditions of zero initial energy storage (zero state response), when a unit step forcing function is applied at the input at t = 0. We introduced the impulse response first in this chapter because it is generally easier to obtain than the unit step response. Remember that the impulse response for t > 0 is just the natural response, while the step response contains both forced and natural components' We have already solved for the step response of first order systems in Chapter 9. The fmal example of Chapter 9 found the response of a second order system to a step function driver. However, this result was not the system's "step response" because the driving force was not a unit step, and because the system had initial energy storage as indicated by the given initial conditions. Nevertheless, the method used to solve that example can also be used to find unit step response:

1. Use roots of the characteristic equation to find the form of the natural response, yn(t).
2. Find the forced response, yf(t) (usually a constant).
3. Add yn(t) and yf(t) together to get the complete response, y(t), then use initial conditions to evaluate constants.

(A second approach is to find the unit impulse response and make use of the fact that the unit step response is the integral of the unit impulse response.)

Example: Unit step response:

We again use the system of the final Chapter 9 example, but this time we want the unit step response.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = u(t) \tag{38}$$

Solution:

1. Find the form of the natural response:

$s^2 + 4s + 5 = 0$ has roots $s_1, s_2 = -2 \pm j$. Therefore, the natural response is

$$y_n = A_1 e^{(-2-j)t} + A_2 e^{(-2+j)t}$$

2. Find the forced response:

$$y_f = 1/5, \quad t > 0.$$

3. Determine the constants of integration.
The complete response is the sum of the natural and forced responses.

$$y(t) = y_n + y_f = A_1 e^{(-2-j)t} + A_2 e^{(-2+j)t} + \frac{1}{5}, \quad t > 0 \tag{39}$$

Since we are finding the unit step response, we know the values of y and dy/dt at t = 0' ar both zero. What are their values at t = 0+? A look at Equation 39 indicates that dy/dt and y canno be step functions at t = 0 (because other terms would then be impulses). This means the values o: dy/dt and y at t = 0+ must be the same as they were at t = 0', (which was zero).

y(0) -O=AI +A2+ 1/5

$$\frac{dy}{dt}(0) = 0 = (-2 - j) A_1 + (-2 + j) A_2$$

solution of these equations yields A, = (-1 + 2j)/10 and A2 = (-1 - 2j)/10. Substitution into Equation 40 and application of Euler's identity then provides the unit step response (we designate unit step response by ys):

$$y_s = \frac{e^{-2t}}{5}\left[\cos t + 2 \sin t\right] u(t)$$

Chapter 10 Problems

1' Find the unit step and impulse responses of a system whose equation is:

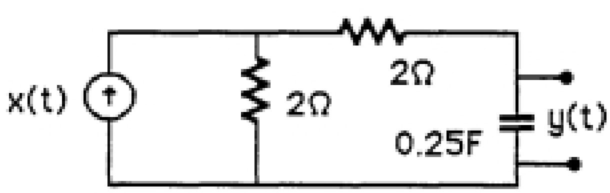
$$x(t) = \frac{dy}{dt} + 2y \int y\,dt$$

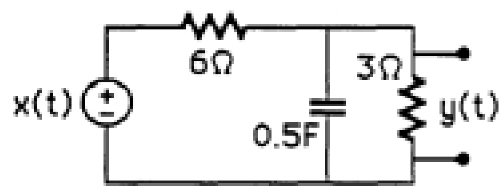
2. For each system whose equation is given below, determine:
   (a) The unit step response
   (b) The unit impulse response.

$$2\frac{d^2y}{dt^2} + 98y = x(t) \qquad \frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 18y = x(t)$$

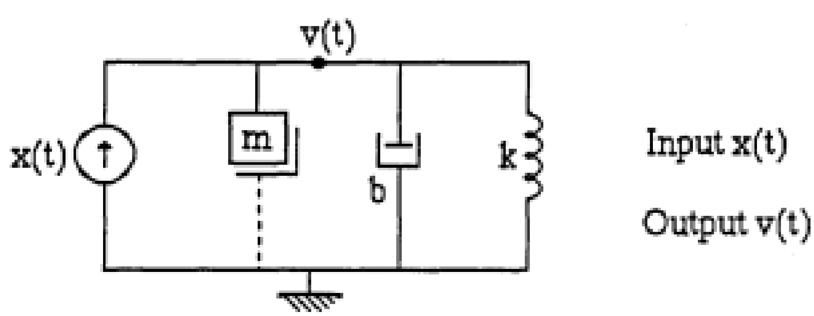
$$4\frac{d^2y}{dt^2} + 40\frac{dy}{dt} + 100y = x(t) \qquad 2\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = x(t)$$

3. Find the impulse response of each of the following systems:





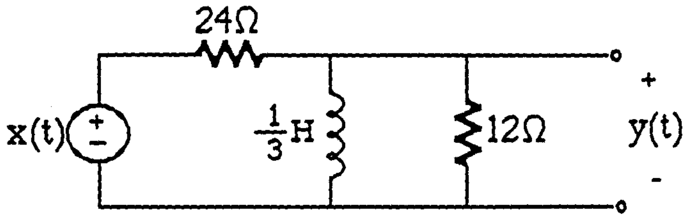
4. If m = 2, b = 12, k = 16, find this system's impulse response.



6. Find the step and impulse response for each system below:

$$x(t) = \frac{1}{2}\frac{dy}{dt} + 2y + 6\int y\,dt$$

(b)



Systems Chapter 11 Study Guide

Convolution

A. Concepts Addressed By This Topic

1. The convolution integral.
2. Convolution using step or impulse response.
3. Computer methods of obtaining convolution.
4' Graphical meaning of convolution.

B. Introduction

The methods of system analysis we have encountered this far require us to obtain a differential equation or other mathematical expression of the system's input-output relation, as well as a mathematical expression of the forcing function (input). In many situations it is not possible to get either a set of differential equations completely describing the system, or an analytic expression for the forcing function. In such cases, the convolution integral becomes a useful tool because it permits calculation of the system's response to a given input waveform without using the system equation or knowing an analytic expression for the input fiinction.

The convolution integral does require a knowledge of the system's impulse response. While we have seen how to find the impulse reponse analytically when the system's differential equation is known, convolution may also be applied when this equation cannot be found if certain physical tests can be performed on the system. If an input "spike" approaching an impulse is applied to the system and its output is recorded, this information can be used by a computer to solve the convolution integral to determine y(t) for any speed or measured x(t). Note: this technique is good only for linear systems.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Given analytic expressions for a system's impulse response h(t) or step response yu(t), use the convolution integral to find the output of the system for any arbitrary input function, x(t).

2. Given sketches of the input function and the impulse response vs. time, sketch the general shape of the resulting output function.

3. Given the input function and impulse response in sampled (digital) forth, use the computer to determine the output waveform.

D. Study Procedure

Review Chapter 10 on impulse response.
Read Chapter 11.
Additional material can be found in reference 13.

E4 Systems
Chapter 11
Convolution

As we saw in Figures 2 and 3 of Chapter 10, a system's response to a short duration pulse of any shape approximates its impulse response. Suppose we approximate a system's input function, x(t), with many rectangular pulses as shown in Figure 1. If the width of each of these pulses, $\Delta\lambda$, is small compared to the system's time constant, each rectangle appears to the system as an impulse function with area proportional to $x(\lambda_i)$. Each of these input pulses will produce, in turn, a separate impulse response at the output. The output function, y(t) will then be the sum of all of these impulse responses with appropriate time delays.

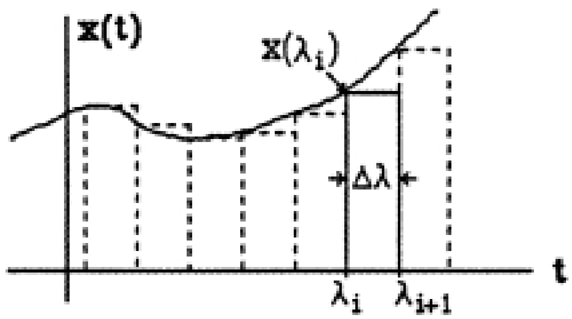


Figure 1 Approximation of x(t) by rectangular pulses

Figure 1 shows x(t) approximated by rectangular pulses of width $\Delta\lambda$ and amplitude $x(\lambda_i)$. A representative one of these pulses is shown extending from $t = \lambda_i$ to $t = \lambda_{i+1}$. Using step functions, we can write an expression for this pulse: $x(\lambda_i)\,[\,u(t - \lambda_i) - u(t - \lambda_{i+1})]$. By summing these pulses everywhere along the t axis, an approximation for x(t) can be written:

$$x(t) \approx \sum_{i=-\infty}^{+\infty} x(\lambda_i)\left[u(t - \lambda_i) - u(t - \lambda_{i+1})\right] \tag{1}$$

Multiply and divide through Equation 1 by $\Delta\lambda$, giving:

$$x(t) = \sum_{i=-\infty}^{+\infty} x(\lambda_i)\left[\frac{u(t - \lambda_i) - u(t - \lambda_{i+1})}{\Delta\lambda}\right]\Delta\lambda \tag{2}$$

As $\Delta\lambda$ gets very small, the quantity in the brackets of Equation 2 becomes a very tall, very narrow rectangle. Its height is $1/\Delta\lambda$ and its width is $\Delta\lambda$. In other words, the rectangle becomes a unit impulse: $\delta(t - \lambda_i)$.

Substituting the impulse function for the brackets in Equation 2 yields

$$x(t) \approx \sum_{i=-\infty}^{+\infty} x(\lambda_i)\,\delta(t - \lambda_i)\,\Delta\lambda \tag{3}$$

The area under an impulse function is given by its multiplier. Therefore, the quantity under the summation sign in Equation 3 is a unit impulse of area $x(\lambda_i)\Delta\lambda$. Equation 3 says that any x(t) can be approximated by a series of closely spaced impulses, each of area $x(\lambda_i)\Delta\lambda$. Note that by "approximated" we mean that if the impulse string is used in place of its corresponding x(t) as the input to a linear system, the output produced will not be significantly different.

Each impulse at the input produces an impulse response at the output, equal to $x(\lambda_i)\,h(t - \lambda_i)\,\Delta\lambda$, where h(t) is the system's impulse response. Therefore, when all impulses are applied, y(t) is approximated by the sum of the resulting impulse responses:

$$y(t) = \sum_{i=-\infty}^{+\infty} x(\lambda_i)\,h(t - \lambda_i)\,\Delta\lambda \tag{4}$$

which, in the limit as $\Delta\lambda \to 0$, becomes

$$\boxed{y(t) = \int_{i=-\infty}^{+\infty} x(\lambda)\,h(t - \lambda)\,d\lambda} \tag{5}$$

The integral of Equation 5 is known as the convolution integral. By a substitution of variables, a second form can be found. Let $\tau = t - \lambda$. Then $d\tau = -\,d\lambda$ and $\lambda = t - \tau$. Substituting, $x(\lambda)\,h(t - \lambda)\,d\lambda$ becomes $x(t - \tau)\,h(\tau)\,(-d\tau)$. Since the integral is between $\pm\infty$, the negative sign on $d\tau$ may be neglected. Using the symbol λ instead of τ then gives:

$$\boxed{y(t) = \int_{i=-\infty}^{+\infty} x\,(t - \lambda)\;h\,(\lambda)\,d\lambda} \tag{6}$$

If the impulse response of a system is known, either Equation 5 or 6 can be used to find the system's response to any given input function, x(t). If the impulse response is not known in analytic form, it may be possible to apply an impulse-like spike at the input and record the waveform produced at the output' Equations 5 or 6 may then be solved numerically.

The above integrals can also be manipulated to provide an expression for the output in terms of the step response and the input.

From (6), let $u = x(t-\lambda)$, and $dv = h(\lambda)d\lambda$. Then,

$$du = -\dot{x}(t-\lambda)\,d\lambda, \qquad v = \int h(\lambda)\,d\lambda$$

The integral of the unit impulse reponse is the unit step response. Call the unit step response $y_u(\lambda)$. Then, $v = y_u(\lambda)$. Combining these u and v values into an integration by parts yields:

$$y(t) = x(t-\lambda)\,y_u(\lambda)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} y_u(\lambda)\,\dot{x}(t-\lambda)\,d\lambda \qquad (7)$$

Since the step response is zero at $\lambda = -\infty$, and assuming $x(-\infty) = 0$, we have:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} y_u(\lambda)\,\dot{x}(t-\lambda)\,d\lambda} \qquad (8)$$

A rearrangement of the variables can convert this to:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} y_u(t-\lambda)\,\dot{x}(\lambda)\,d\lambda} \qquad (9)$$

Equations 8 and 9 are sometimes more convenient than 5 or 6 when the step response is known, or in practical cases where the impulse response is difficult to generate.

Convolution by computer:

As has been mentioned, convolution has the advantage that analytic functions are not really necessary to compute the system's response to an arbitrary input. For example, suppose the system is hit with an impulse and its output is monitored and recorded. Let's say the resulting output is a wave such as shown as h(t) in Figure 2. Suppose that at every Δt step in time, we sample the waveform, and store the resulting list of m values, h[i] for i=1 to m, in a computer file.

(Such sampling is easily done by a common device called an analog to digital (A to D) converter.) We want to determine this system's response to an arbitrary forcing function, x(t), which is also plotted and sampled, yielding a second list of n values, x[i] for i = 1 to n.

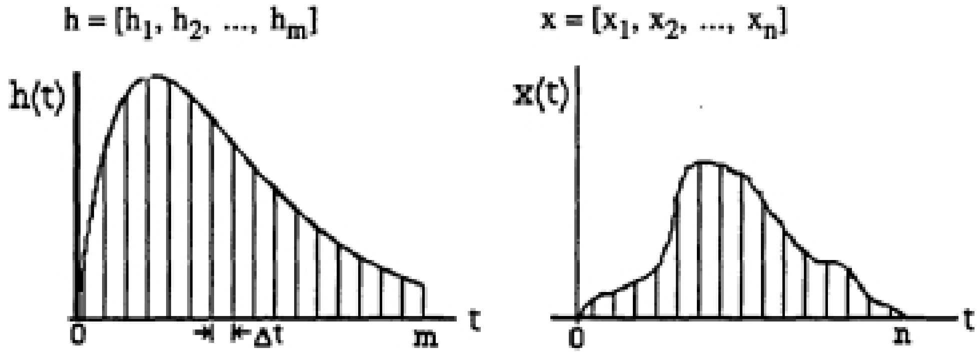


Figure 2 h(t) and x(t) sampled every Δt seconds.

We decide to use Equation 6. This means x(t) must be reversed and stepped past h(t). At each step, the all matched values must be multiplied and summed. This is tedious work for humans, but easily done by a computer. A Pascal solution for the convolution of x and h is given below (assume Δt=1 for convenience):

Pascal Convolution Procedure:

```
{h[i] is an array of m values, x[i] is an array of n values)
        ( a is an integer used to index through the values)

        for t:= 1 to (m+n) do
        begin
                sum:=0;          { sum gives the value of y(t) at each step }
                a:=l;            { a steps through from a= 1 to a=t)
                while (a<=n) and ((t-a)>=I) do
                        begin
                                sum:=sum + x[t-a]*h[a]; { add the newest product)
                                a:=a+1;                          {index to the next pair}
                        end;    I while. Integral to this t is complete)
        { here, write to file, plot, etc' value of "sum")
        end; {for.          Increment t and integrate again}
```

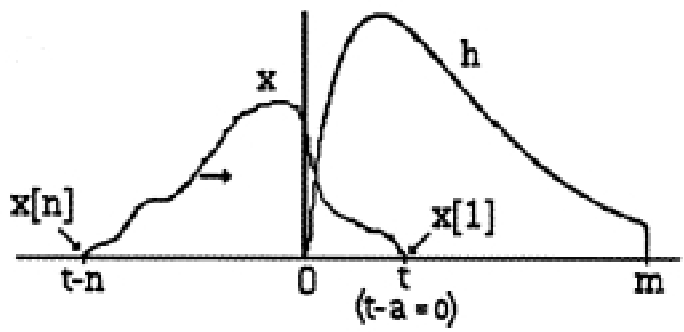


Figure 3 Convolution process as x steps past h

Example 1

A system has impulse response $h(t) = 10\, e^{-5t}\, u(t)$. Find its response to a step function input, $x(t) = u(t)$, by convolution.

Solution:

Step 1: Set up the convolution integral. Use the form of either Equation 5 or Equation 6. (We choose to use λ for the argument of the step function, and $t-\lambda$ as the argument of the impulse response as in Equation 5.)

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda = \int_{-\infty}^{+\infty} u(\lambda)\, 10\, e^{-5(t-\lambda)}\, u(t-\lambda)\, d\lambda \tag{10}$$

Step 2: Adjust the limits. Generally, step functions will accompany the functions being convolved. These should be eliminated within the integrand by proper adjustment of the limits of integration. In the case of this example, the lower limit can be reset to zero because the $u(\lambda)$ factor is zero for $\lambda < 0$. The upper limit can be set to t because $u(t-\lambda)$ is zero for $\lambda > t$. After adjustment of the limits, the step functions can be dropped from the integrand.

$$y(t) = \int_0^t 10\, e^{-5t} e^{5\lambda}\, d\lambda = 10\, e^{-5t} \int_0^t e^{5\lambda}\, d\lambda = 2\ e^{-5t}\ e^{5\lambda}\Big|_0^t \tag{11}$$

Substitution of the limits now yields y(t), good for t>0. Since we know this system was zero for t<0, we can use a step function to describe it.

$$y(t) = 2\,[\,1 - e^{-5t}\,]\, u(t) \tag{12}$$

An examination of Equation 13 will show that we could also have obtained the result of Equation 12 (which is the step response of the system), by integrating the impulse response.

$$2\left[1 - e^{-5t}\right] = \int_0^t 10\, e^{-5t}\, dt \tag{13}$$

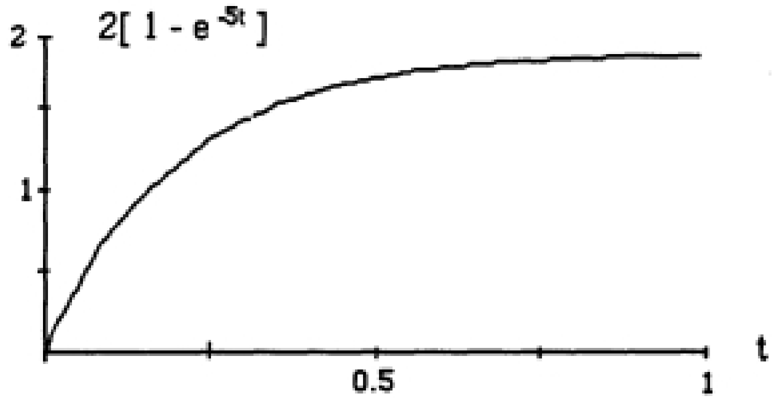


Figure 4 Response of system of Example 1 to a step input

Of course, the evaluation of the integrals of Equation 10 or 11 may also be computed using Maple. However, present versions of Maple do not handle step functions very well. Therefore, the set up of the integral and the adjustment of the limits must be done before Maple's int function is used. (In other words, apply Maple to Equation 11, not to 10.) After the integration has been done, the proper step function multiplier must also be inserted by hand. The Maple commands to perform the integration of Equation 11 are given in bold type below:

```
int(10*exp(-5*t)*exp(5*L),L=0..t);
                                        2 exp(- 5 t) exp(5 t) - 2 exp(- 5 t)

simplify(");
                                                  2 - 2 exp(- 5 t)
```

Example 2

Find the output y(t) if a signal $x(t) = e^{-2t}\,u(t)$ is applied to a system whose impulse response, $h(t) = 2\,e^{-2t}\,u(t)$.

Solution:

Step 1: Set up the convolution integral between infinite limits:

$$y(t) = \int_{-\infty}^{+\infty} 2\, e^{-2\lambda}\, u(\lambda)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda \tag{14}$$

Step 2: Adjust limits and remove step functions from integrand:

$$y(t) = \int_0^t 2\, e^{-2\lambda}\, e^{-2(t-\lambda)}\, d\lambda = 2\int_0^t e^{-2t}\, d\lambda = 2\, e^{-2t}\lambda\Big|_0^t$$

Step 3: Insert proper step function into the result:

$$y(t) = 2\, t\, e^{-2t}\, u(t) \tag{16}$$

Graphical convolution

A quick indication of the type of output wave to expect can sometimes be achieved by a graphical approach to the convolution integral. Note that the integral always involves replacing t by t - λ in one of the functions. For example, a function x(t) is plotted vs. t in the left panel of Figure 5. If we now plot the same x function on the λ axis as a function of (t - λ), it has the same shape, but is reversed, because of the minus sign on λ. In the left panel, we see that x(0) is zero. On the right, the x(0) value occurs when λ = t, since this function is x(t - λ). (t is not a variable on the right panel; it is just a parameter.) On the left, the function rises as a ramp until t = a, at which time the value of x is x(a) = 4. Where does this value of the function occur on the right? The function on the right will be x(a) when (t - λ) = a, or when λ = t - a. Thus the effect of changing the argument from t to t - λ is a reversal of the wave and a translation of the old origin to λ = t.

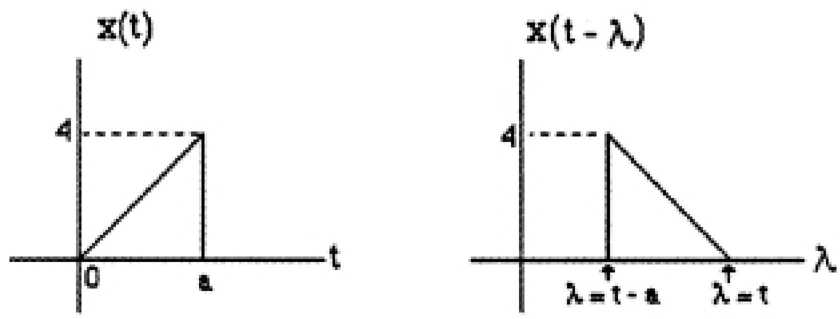


Figure 5 Effect of replacing argument t by t - l.

The convolution integral tells us to reverse one of the two functions (x or h), and integrate the product of the two for each t value for which we want to know y. Since we generally want y(t) for all t from 0 to ∞, this means the integral must be recomputed for each t value from 0 to ∞. Graphically, we can think of this operation as the translation of both plots to the λ axis, with one graph reversed. As t is changed, the reversed plot (the one using the t - λ argument) moves to the right. The other plot (the λ argument function) stays put. For each t, the two functions must be multiplied point by point and the area under their product curve evaluated. This area is y(t).

As an example, let's assume h(t) is a rectangular pulse such as given in Figure 6.

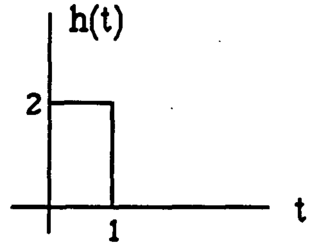


Figure 6 Example impulse response

We will use the x(t) function defined in Figure 5 as a driving force. The convolution integral (using the form of Equation 6) is

$$y(t) = \int_{-\infty}^{+\infty} x(t - \lambda)\, h(\lambda)\; d\lambda$$

The x function on the λ axis appears as shown on the right of Figure 5. Its leading edge is at the point $\lambda = t$. This shape is shown for two t values (0 and 4) in Figure 7, along with $h(\lambda)$.

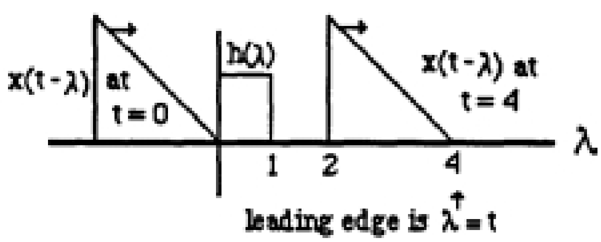


Figure 7 h(,) and x(t - X) shown at t = 0 and t = 4

The convolution integrand is the product of the two functions. Therefore, it results in a non-zero outcome (area under the product) only when the functions overlap. Figure 7 indicates that there is no overlap before t = 0. The two shapes then overlap until t = 3, at which time the trailing edge of the x function is at $\lambda = 1$. So from t = 0 to t = 3, we must step the x function past the h function and at each step, multiply the two functions together and find the area under this product.

In Figure 8, the functions just discussed are shown on the λ axis, with smaller steps taken as t goes from 0 to 3. At each step in t, the x function moves to the right. (Its leading edge is always at $\lambda = t$.) For each step, the product of the two curves is graphed and the area under the product is shown shaded. A plot of this area vs. t, shown in Figure 9, is y(t), the result of the convolution of x(t) and h(t). Note that the result plotted in Figure 9 is not the area of the overlap, but the area under the product of the two functions.

The example illustrated in Figures 8 and 9 can be worked out in reasonable detail because the x and h functions are nice geometric shapes. However, such things as times at which y(t) begins and ends, and estimates of maximum value or time of maximum can often be done by inspection of the h and x waveforms, without calculation.

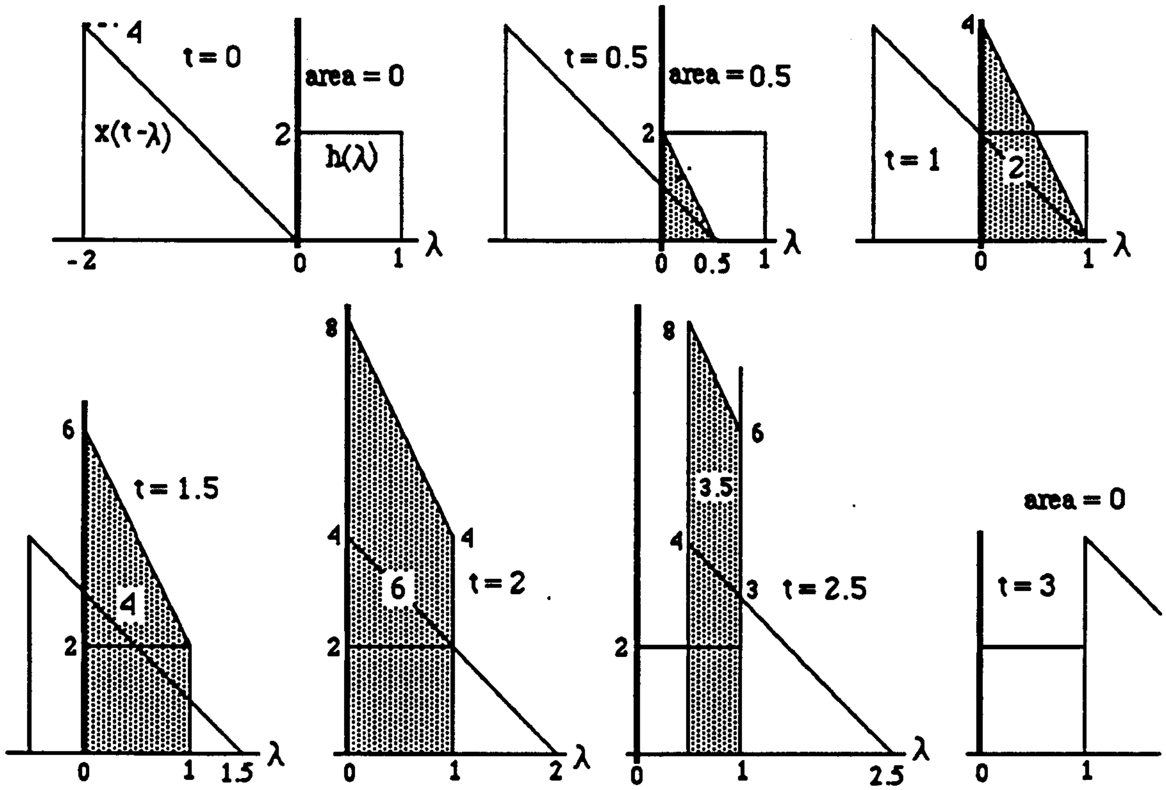


Figure 8
Successive time steps showing x(t - λ), h(λ), and the area under their product.

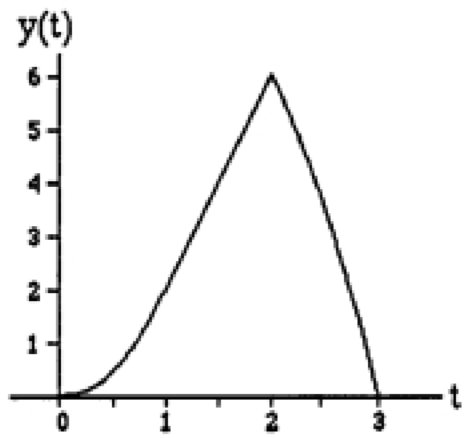


Figure 9 y(t): Plot of the areas of Figure 8 vs t

1.Find y(t) by convolution if

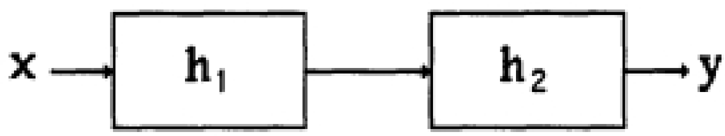
(a) $x(t) = e^{-t}\,u(t)$, $h(t) = e^{-2t}\,u(t)$.

(b) $x(t) = 2e^{-2t}\,u(t)$, $h(t) = e^{-2t}\,u(t-1)$.

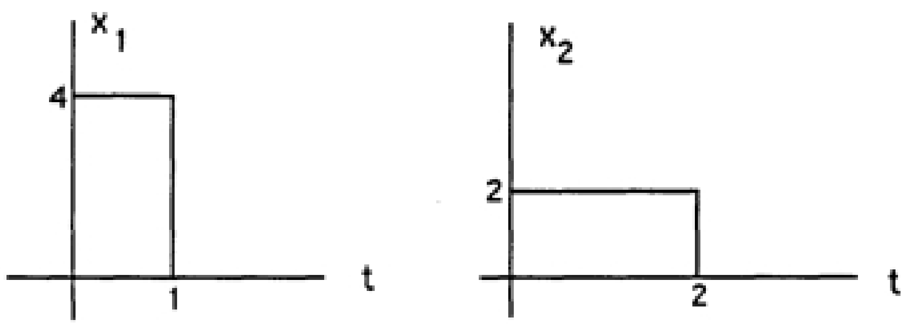
(c) $x(t) = e^{-2t}\,u(t)$, $h(t) = \delta(t) - e^{-3t}\,u(t)$

(d) Unit step response is $4(1 - e^{-5t})\,u(t)$, input is $x(t) = 3e^{-5t}\,u(t)$.

2. Show that for systems in cascade, the overall impulse response (y below for x equal to a unit an impulse) is hl (t)*h2(t), where * means convolution.

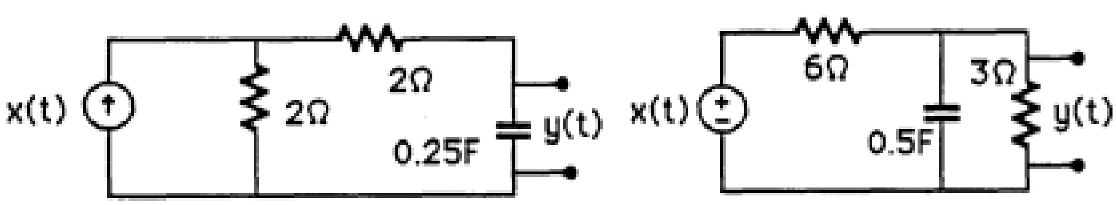


3. Graphically determine the convolution of the two functions below. Label all important values.



4. A system has an impulse response $h(t) = e^{-2t}\,u(t)$. Find its response to the input $x(t) = (e^{-2t} + e^{-3t})\,u(t)$ by convolution.

5. Using Maple or other computer help, determine the response of each of the systems below to a half cycle of x(t) = sin t (from t = 0 to π). The impulse responses of these systems were solved as part of an earlier homework assignment.



6. The system below has an impulse response $h(t) = 6\,e^{-2t}\,u(t)$.

If the input $x(t) = 3\,e^{-4t}\,u(t)$ is applied to this system, what is y(t)?

Systems Chester 12 Study Guide

- System Function

A. Concepts Addressed By This Topic

1. System function, H(s).
2. Impedance, Z(s), Admittance, Y(s).
3. s-plane, poles and zeros.
4. System response from system function.

B. Introduction

Forms of the exponential e^{st} can be used to represent most waveforms which appear in linear systems. Constant and sinusoidal forms are often used as driving forces. Exponential decay functions often appear as part of the natural response. It is also possible to represent virtually any periodic waveform which might appear in physical systems by summing an infinite series of exponentials of the form $A_n e^{snt}$ (Fourier series), where the integer n represents harmonic number.

When an exponential forcing function is applied to a linear system, the forced response is identical in form to the forcing function, differing only by a multiplicative constant. The multiplying constant is a function of complex frequency, s, and is given the symbol H (s). H (s) therefore is the ratio of the forced output exponential function to input exponential forcing function. When output and input are measured at the same port, H (s) is called a <u>driving point</u> function. When these signals are measured at two different ports, H (s) is called a <u>tranfer</u> function. If the ratio is between an effort variable (numerator) and a flow variable (denominator), H (s) may be called *impedance , Z(s).* The inverse of impedance is admittance, Y(s).

The denominator roots of H(s) are values of s for which the numerator (the response) can be non-zero even for a zero denominator (forcing function). A non-zero response to a zero forcing function has already been defined as the system's natural response. Values of s which make the denominator zero are called poles. The value of the poles of a system are then the natural frequencies of the system. Numerator roots represent driving function frequencies for which there is zero forced response.

The forced response to any forcing function which can be represented by an exponential can be obtained by multiplication of the forcing function by H (sf).

C. **Instructional Objectives**

A student who masters this material will be able to

1. Given the differential equation of a linear system, determine the system function, H(s).
2. Given the impulse response, h(t), of a linear system, determine H(s).
3. Given the <u>graph</u> a linear system, determine H (s) using impedance concepts.
4. Plot poles and zeros of H(s) in the s plane.
5. Simplify system graphs by using loop/node methods, series/parallel combinations, Thevenin or Norton equivalent systems, etc.
6. Given an exponential forcing function, use H (sf) to determine the forced response, yf.
7. Use the poles of H(s) to find the form of the natural response of a system.
8. Combine results of objectives 6 and 7 with initial conditions to find complete response.

D. Study Procedure

Read Chapter 12' Additional material can be found in Reference 11, Chapter 13, and also references 1 and 13.

Chapter 12
System Function

1 The exponential function as a common system signal

Consider the general function

$$x_e(t) = A\, e^{\sigma t} \cos(\omega t + \phi) \tag{1}$$

This function can be used to represent many signals which occur naturally in systems or are used as system input signals. The parameters σ, ω, and ϕ can be adjusted to produce a constant, a sinusoid in any phase position, an exponential decay, and other waveforms of the type shown in Figure 1. (Remember, $s = \sigma + j\omega$):

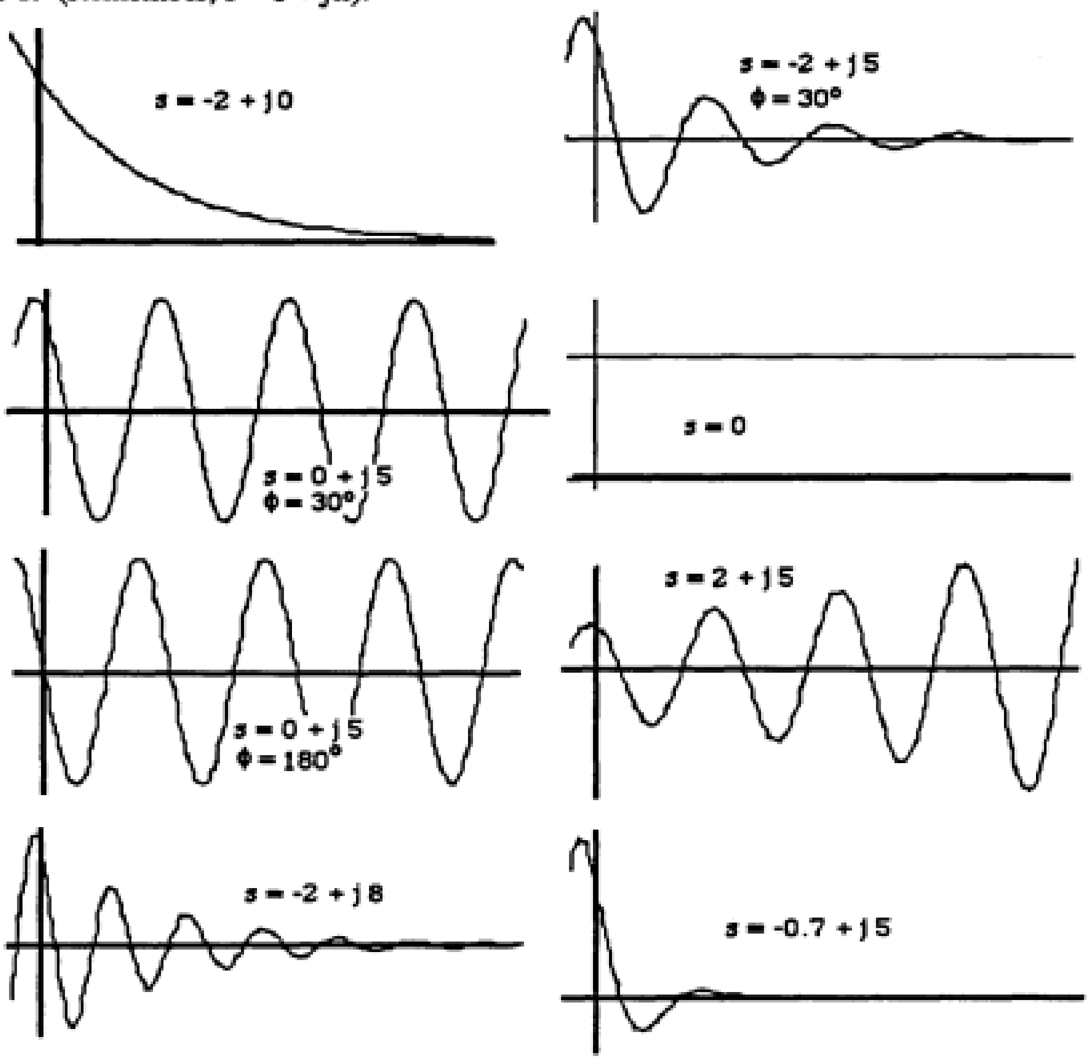


Figure 1 Waveforms Represented by Equation 1

The exponential function $x_e(t)$ can be represented by a rotating phasor. We have already seen that the function $x(t) = A\cos(\omega t + \phi)$ is the real part of a rotating phasor, $\mathbf{A}\ e^{j\omega t}$, where $\mathbf{A} = A\ e^{j\phi}$ as shown in Figure 2.

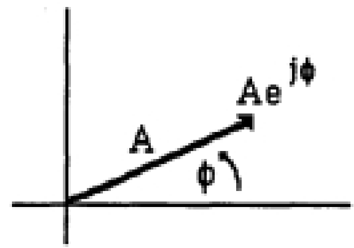


Figure 2 Phasor $\mathbf{A} = A\ e^{j\phi}$

If we multiply $\mathbf{A}$ by $e^{j\omega t}$, the line of Figure 2 rotates at angular velocity, ω. If, in addition, we multiply by $e^{\sigma t}$, the line length changes as it rotates. For example, $\mathbf{A}\ (e^{\sigma t})(e^{j\omega t})$, for negative σ, traces a spiral as shown in Figure 3.

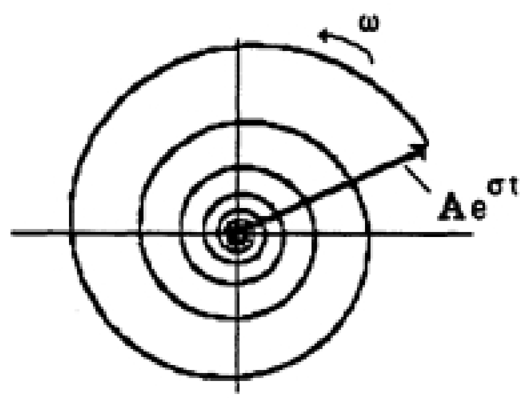


Figure 3 Locus of end point of phasor $\mathbf{A}\ (e^{\sigma t})(e^{j\omega t})$

The projection of a phasor of this type on the real axis is $A\ e^{\sigma t}\cos(\omega t + \phi)$, one form of which is plotted in Figure 4.

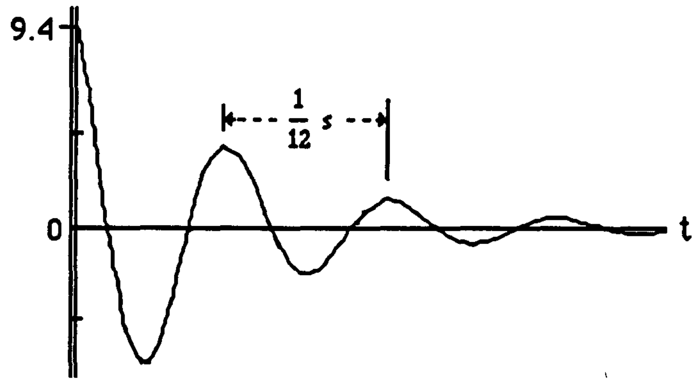


Figure 4 Plot of $10\ e^{-2t} \cos(12t + 20^o)$

Therefore, our general function $x_e(t) = A\ e^{\sigma t} \cos(\omega t + \phi)$ can be represented by a rotating phasor. Remember that $\mathbf{A} = A\ e^{j\phi}$. Therefore, the phasor representation of x(t) is $\mathbf{A}\ (e^{\sigma t})(e^{j\omega t}) = \mathbf{A}\ e^{(\sigma + j\omega)t} = \mathbf{A}\ e^{st}$. Here we have substituted $s = \sigma + j\omega$. Note that σ and ω, and therefore, s, have the same dimensions, seconds^{-1}. (The dimension of ω is often called radians/sec and that of σ may be called nepers/sec. However, radians and nepers are dimensionless.) σ and ω may be referred to as frequencies, and s is called "complex frequency". The real part of the phasor is $x_e(t)$.

$$x_e(t) = A\ e^{\sigma t} \cos(\omega t + \phi) = \mathrm{Re}[\ \mathbf{A}\ e^{st}\ ] \tag{2}$$

Therefore, Re[$\mathbf{A}\ e^{st}$] can represent any of the waveforms of Figure 1. We can use the complete $\mathbf{A}\ e^{st}$ in all mathematical operations involving the function of Equation 1 as long as we use only the real part of the result. The need to convert to the real part arises only in the case of sinusoids ($\omega \neq 0$).

2. Exponential Function Applied to a Linear System

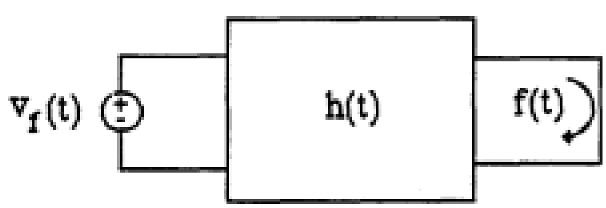


Figure 5 Exponential forcing function applied by A-type source

Figure 5 shows a linear system with forcing function $v_f(t)$ applied. The response (output) is defined to be one of the system's T variables, labeled f in the figure. The impulse response of this system is h(t). That is, if $v_f = \delta(t)$, then f = h(t). Suppose we apply an input signal represented by the rotating phasor $\mathbf{V}_f(t) = \mathbf{V}\, e^{s_f t}$. s_f stands for the complex freqency of the driving force $(\sigma_f + j\omega_f)$. Using the convolution integral, we can find the forced response, $f_f(t)$, produced by v_f. Use $v_f(t - \lambda)$ and $h(\lambda)$:

$$\mathbf{F}(t) = \int_{-\infty}^{+\infty} \mathbf{V}\, e^{s_f(t-\lambda)}\, h(\lambda)\, d\lambda = \mathbf{V}\, e^{s_f t} \int_{-\infty}^{+\infty} e^{-s_f \lambda}\, h(\lambda)\, d\lambda \tag{3}$$

F(t) is used because we will get the (complex) phasor representation of ff(t) from this operation. The integral part of the right side of Equation 3 is not a function of time, but a constant function of sf. We will define the result of the integration H(sf), the system *function.* Since h is the impulse response, then for causal systems, the lower limit on the integral can be zero. Then, in general, for any frequency s,

$$\mathbf{H}(s) = \int_{0}^{\infty} e^{-s\lambda}\, h(\lambda)\, d\lambda \tag{4}$$

This operation on $h(\lambda)$ to get $\mathbf{H}(s)$ is called the Laplace transform. That is, $\mathbf{H}(s)$ is the Laplace transform of $h(\lambda)$. We will work with the Laplace transform more extensively later in this course. For now, the significance of Equations 3 and 4 is that the forced response, $f_f(t)$ is seen to be just a constant, $\mathbf{H}(s_f)$, times the forcing function, $v_f(t)$. That is,

f~t) = H(sf) vt(t) (5)

Therefore, if H (s) is known, the steady state response to any exponential forcing function can be found by evaluating H(sf) where sf is the forcing function's frequency. Our next task is to investigate methods of determining H (s) for a given system.

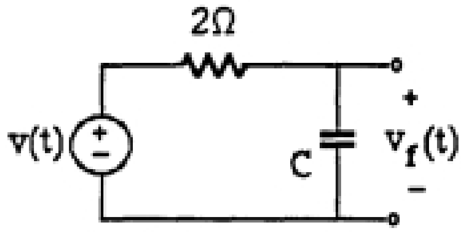
3. System Function H(s)

We have seen that for the case of an exponential driving function of frequency s f, the system function, H(sf), is a constant which relates the forced component of the response to the forcing function. H (s) actually has more general use in the solution of systems, which we will fimd when we study Laplace transform techniques later in this course. For now, Equation 5 may be used to expand our ability to find the forced component of system response. (Only step and other singularity function drivers have been previously considered.) To use Equation 5, we must first determine H (s). One way to obtain H (s) is from its definition, Equation 4. However, in many cases, other methods are available to determine H (s) more directly from its

3.1 Determination of H(s) using Equation 4

If the impulse response is known or is easily found, a direct approach using Equation 4 may be used to find **H**(s).

Example: In the system below, the output is defined to be the voltage across the capacitor. Find the forced response, v_f, if $v(t) = 4e^{-2t}\,u(t)$.



Solution: The impulse response for this system is

$$h(t) = \frac{1}{RC}\,e^{-\frac{1}{RC}t}\,u(t) = \frac{1}{2}\,e^{-\frac{1}{2}t}\,u(t)$$

From Equation 4

$$H(s) = \int_0^\infty e^{-st}\left(\frac{1}{2}e^{-\frac{1}{2}t}\,u(t)\right)dt = \frac{1}{2}\int_0^\infty e^{-(s+\frac{1}{2})t}\,dt$$

which yields

$$H(s) = \frac{\frac{1}{2}}{s+\frac{1}{2}}$$

For the problem above, $s_f = -2$. $H(-2) = -1/3$. This is the ratio of the forced part of the output to the input exponential, according to Equation 5. That is,

$$v_f(t) = -\frac{1}{3}\left[\,4\,e^{-2t}\,u(t)\,\right]$$

3.2 **Determination of H(s) from the system differential equation**

In most cases where a system diagram is given, the method of Section 3.1 is not convenient since it first requires a computation of the impulse response. H (s) can be determined directly from the system differential equation which can be found directly from the system diagram. Given a general system differential equation,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y =$$

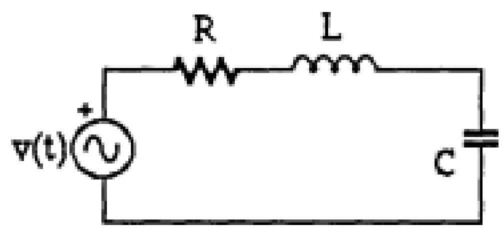
$$b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x \qquad (6)$$

Assume our forcing function x(t) can be represented by an exponential phasor $\mathbf{X}e^{s_f t}$. Then according to the theory we developed in the previous section, the forced part of the response is $\mathbf{Y}e^{s_f t}$. These functions must satisfy Equation 6. Plugging them in and carrying out the indicated differentiation yields:

$$(a_n s_f^n + a_{n-1} s_f^{n-1} + \dots + a_1 s_f + a_0)\mathbf{Y} e^{st} = (b_m s_f + b_{m-1} s_f^{m-1} + \dots + b_0)\mathbf{X} e^{st} \qquad (7)$$

Then, $$\frac{\text{forced response}}{\text{input}} = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{(b_m s_f^m + b_{m-1} s_f^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)} = \mathbf{H}(s) \qquad (8)$$

Example: Given the series RLC system below, for which the output is defined to be the current, find $\mathbf{H}(s)$ and the forced component of the current if the input voltage is v(t) = 10 cos 3t and R = 4Ω, L = 1 H, C = 1/18 F.



Solution: The system differential equation is

$$v(t) = L\frac{di}{dt} + R\,i + \frac{1}{C}\int_0^t i\,dt + v_C(0)$$

Differentiate to clear of integrals:

$$\frac{dv}{dt} = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\,i$$

From Equation 7, we see that this corresponds to

$$s_f V = s_f^2 L I + s_f R I + \frac{1}{C} I$$

where V is the phasor representation of the forcing function, of (t) = 10 cos 3t, and I is the phasor representation of the forced response, if(t). Then H (sf) can be found as the ratio of the phasor representing forced response (1), to the phasor representing the forcing function (V).

$$\frac{\mathbf{I}}{\mathbf{V}} = \frac{s_f C}{s_f^2 LC + s_f CR + 1} = \frac{s_f}{s_f^2 + 4 s_f + 18} = \mathbf{H}(s_f)$$

The frequency applied by v f(t) is sf = j 3. Substitute this into the above equation to

$$\mathbf{H}(j3) = 0.2 \angle 37^o$$

H(j3) multiplies the input phasor to produce the output phasor. That is, the output magnitude is the input magnitude (10 volts) multiplied by 0.2, and the output angle is the input phase angle (0^o), shifted by 37^o. Therefore, the forced response is

$$i_f(t) = 2 \cos (3t + 37^o)$$

3.3 The Concepts of Impedance and Admittance

Consider the individual forced responses to exponential functions of our three generalized passive system elements: a generalized capacitance, a generalized resistance, and a generalized inductance. The forcing function can be either an A- or a T-type driver. As we have found, the response to an exponential forcing function represented by $\mathbf{A}_1 e^{s_f t}$ can be represented by another exponential function, $\mathbf{A}_2 e^{s_f t}$ and the ratio of forced output over exponential input, $\mathbf{H}(s_f)$ is a constant. This constant may be given a special name, depending on the type variable found in the numerator and denominator. When the forcing function and the forced response are measured at the same terminals (the same port), $\mathbf{H}(s_f)$ is called a *driving point* function. When the forcing function is applied to one port and the response is measured at a different port, $\mathbf{H}(s_f)$ is called a *transfer function*..

For our generalized system representation, if $\mathbf{H}(s)$ represents a ratio of v (the numerator), to f (the denominator) this ratio is called *impedance*, $\mathbf{Z}(s)$. If $\mathbf{H}(s)$ represents a ratio of f (the numerator), to v (the denominator) this ratio is called *admittance*, $\mathbf{Y}(s)$.

Below are listed the defining equations for the three generalized elements.

$$f = C\frac{dv}{dt}$$ defines generalized capacitance, C

$v = R\,f$ defines generalized resistance, R

$$v = L\frac{df}{dt} \qquad (9)$$ defines generalized inductance, L

Suppose an exponential forcing function is applied to each element above. Find H (s f) in each case. That is, find the ratio of the exponential representation of the response to the exponential representation of the forcing function.

(a) Generalized capacitance:

Forcing function is an exponential v(t) which can be represented by $\mathbf{V} = \mathbf{V}_\phi e^{s_f t}$.

Then, using phasor representations instead of real time functions:

$$\mathbf{F} = C\frac{d}{dt}(\mathbf{V}_\phi e^{s_f t}) = s_f C(\mathbf{V}_\phi e^{s_f t}) = s_f C \mathbf{V}$$

So the impedance (ratio of V to F) of a generalized capacitance is

$$\mathbf{Z}_C(s) = \frac{1}{sC} \qquad (10)$$

(b) Generalized resistance:

For the generalized resistance, $v = R\,f$.

Therefore, for any applied forcing function, the impedance (ratio of v to f) of a resistance is

$$ZR(s) = R \qquad (11)$$

(c) Generalized inductance:

Assume an f(t) forcing function represented by a phasor $\mathbf{F} = \mathbf{F}_\phi e^{s_f t}$.

Then,

$$\mathbf{V} = L\frac{d}{dt}(\mathbf{F}_\phi e^{s_f t}) = s_f L(\mathbf{F}_\phi e^{s_f t}) = s_f L \mathbf{F}$$

From which the impedance of a generalized inductance is seen to be

$$ZL(s) = s\,L$$

In each of these three cases, we could also have found the admittance, Y(s), of each element:

$$Y(s) = \frac{1}{Z(s)} \tag{12}$$

3.4 Combinations of Impedances and/or Admittances to determine H(s)

The impedances or admittances of elements in a system may be combined in a manner similar to resistances in an electric circuit to determine H(s). That is, impedances in series may be added, and admittances in parallel may be added.

Consider the system shown below. An exponential across source applies a driving force which can be represented by the phasor $\mathbf{V} = \mathbf{V}_\phi e^{s_f t}$.

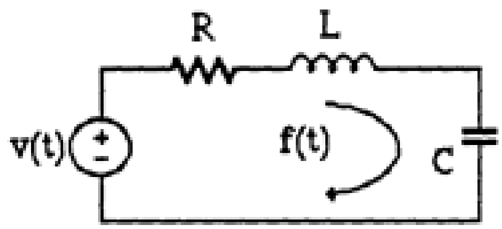


Figure 6 RLC series system

We know that such a source produces a forced response of the form $\mathbf{F}_\phi e^{s_f t}$ in each element. Furthermore, because of the series connection, we know that all elements have the same through variable, f. Using the path rule, we can write

Vin = VR + VL + VC = R F + sL F + (1/sC) F (13)

So the impedance the source works into is Vin/F = Z(s) = R + sL + 1/sC. In other words, we can simply add the impedances of the individual elements, since they are in series, to get the total impedance seen by the source. Also note that Equation 13 could be used to obtain the differential equation for this system by replacing s by d/dt.

For the system of Figure 7, find the input impedance seen by the source:

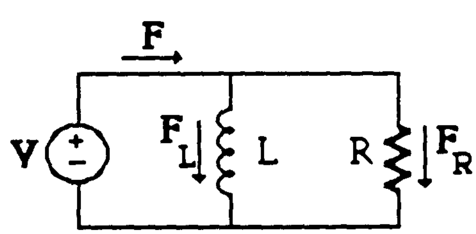


Figure 7 RL system driven by an ideal A-source

FR = V/R and FL = VAL. Therefore,

$$F = F_R + F_L = V(1/R + 1/sL) - VY \qquad (14)$$

The admittance of the parallel elements is the sum of their individual admittances.

Impedance can also be used to determine a transfer function between input and output quantities at different ports to determine forced response:

In the system of Figure 8, the left end of the spring is driven at a velocity $v(t) = e^{-2t} u(t)$. Determine the force exerted on all elements, and the velocity of the mass. Given that m = 1Kg, b = 0.2 Ns/m, and k = 1.2 N/m.

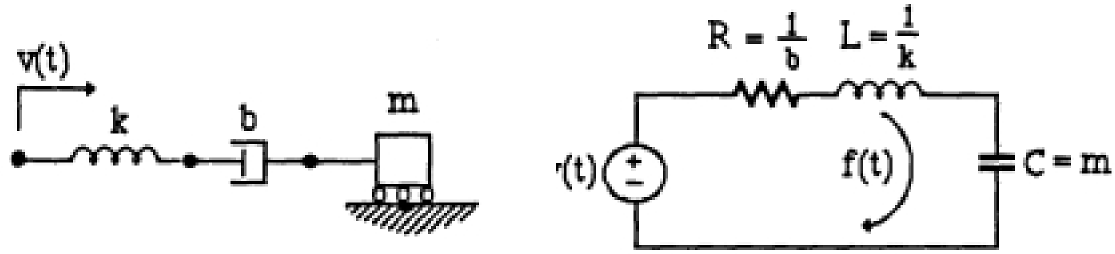


Figure 8 Mechanical system and its generalized

Redraw the system in the generalized equivalent form on the right. The same force flows through each elemenl It can be determined using the impedance seen by the source which is the sum of the impedances of the series elements:

$$Zin(s) = sL + R + 1/sC \qquad (15)$$

For the given forcing function, s f = -2 so find Zin(-2). (Zin(-2) = 2.833.)

Zin is the ratio of Vin (the source) to F. Therefore, the forced part of the force on all elements is $f_f(t) = 0.353\, e^{-2t} u(t)$.

To find the forced component of the velocity of C, we can use the force just determined: $V = (1/sC)\, F$. For $s_f = -2$, this gives $v_f(t) = 0.1765\, e^{-2t}$.

The relation between input and output velocities could have been determined directly from the generalized system diagram if we think of it as a "voltage divider":

$$V_C = \left(\frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} \right) V_S \qquad (16)$$

Plugging in $s_f = -2$ and the values of R, L, and C yields 0.1765 for the expression in the brackets (a transfer function). This tells us that the forced component of the mass' velocity is 0.1765 times the forcing function, or $v_f(t) = 0.1765\, e^{-2t} u(t)$.

3.5 Natural Response From H(s)

We have seen that when the frequency of the exponential forcing function, sf, is substituted into H(s), the result, H(sf), is the ratio of the output forced response to the input forcing function. Suppose the forcing function is zero. That is, what if the A-type or T-type source we are using as a driving force is set to zero. Then H(s) which, by Equation 8, is the output (numerator) over the forcing function (denominator), has a zero denominator. Under this condition, should we expect any response from the system?

Consider the system shown in Figure 9. The 'n u is defined as the current supplied by the current source, and the ou ut is defined to be the voltage which then appears across terminals a-b. The appropriate H(s) (output over input) is Zin. Looking in at terminals a-b we see two parallel branches. One has the impedance ZR=R. The other branch has the impedance ZLC = sL + 1/sC.
The combined impedance of these branches is the product over the sum of their individual impedances. The values of R, L, and C are: R = 3Ω, L = 1H, and C = 0.5F. Using these values of R, L, and C, H(s) is as given in Equation 17.

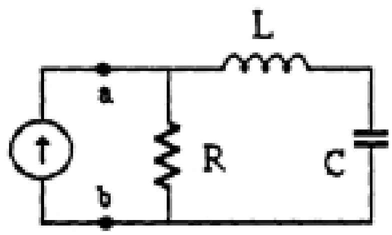


Figure 9 System with current source driver

$$\mathbf{H}(s) = \mathbf{Z}_{in} = \frac{3(s + \frac{2}{s})}{s + 3 + \frac{2}{s}} = 3\ \frac{s^2 + 2}{s^2 + 3s + 2} = 3\ \frac{s^2 + 2}{(s+1)(s+2)} \qquad (17)$$

Now what if our current source is zero? Is it still possible to have a voltage (response) across terminals a-b? A zero current source means open circuit. A look at Figure 10 (zero current source now connected) indicates that a voltage is still possible across the terminals. If the capacitor has an initial charge, for example, it will discharge through the RL path. This will produce a voltage across the resistor (defined as the output) even though the input current is zero.

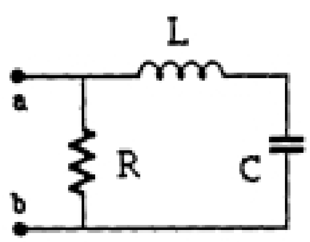


Figure 10 System of Figure 9 with zero current source driver

Can H (s) tell us anything about voltage a-b under this condition? For H (s) to have a zero denominator, s must be -1 or -2. (Values of s which make the denominator of H (s) zero are called poles of the system function.) The frequencies represented by the poles are clearly not coming from the zero driving source. Instead, they represent natural frequencies of the system; frequencies for which an output is possible even with a zero forcing function.

We have seen natural frequencies like these before. They indicate a response of the form

$$v_n(t) = A_1 e^{-t} + A_2 e^{-2t} \tag{18}$$

Therefore, H (s) provides information regarding the natural as well as the forced response of a system. The natural frequencies of the response are represented by the poles of the system function, H(s).

4. Poles and Zeros

For many systems, the system function, **H**(s), is a ratio of two polynomials in s of the form of Equation 8. There are as many roots of the numerator and of the denominator as the order of each respective polynomial. The roots of the numerator polynomial are called *zeros* of **H**(s), and the roots of the denominator are the *poles* of **H**(s). Of course, values of s are frequencies and roots of real polynomials can be real or complex conjugates. Therefore, the zeros and poles of **H**(s) can be plotted in the complex plane. For example, the function of Equation 17 has poles of -2 and -1, and zeros of $\pm j\sqrt{2}$. This function is repeated below. A plot of the poles and zeros of **H**(s) is given in Figure 11.

$$\mathbf{H}(s) = 3\,\frac{s^2+2}{(s+1)(s+2)} \tag{19}$$

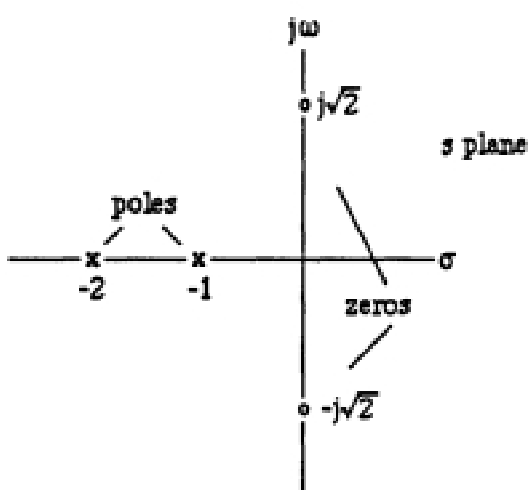


Figure 11 Pole-Zero diagram for H(s) of Equation (19)

Pole significance:

Remember that $\mathbf{H}(s)$ is a fraction representing the output over the input as a function of s. A pole of $\mathbf{H}(s)$ is a frequency which makes the denominator (but not the numerator) of $\mathbf{H}(s)$ equal to zero. Another way of saying this is that at this frequency, an output (numerator) is possible even with a zero input (denominator). The response to a zero input driver is what we have previously defined as the natural response of the system. The poles of the system function therefore are the natural frequencies of the system.

Zero significance:

Zeros are frequencies for which the numerator of $\mathbf{H}(s)$, and therefore $\mathbf{H}(s)$ itself, is zero. Therefore, if we drive a system with a source of frequency equal to one of the zeros of $\mathbf{H}(s)$, the forced component of the response will be zero. For example, if we apply a current source to the system of Figure 9 (Equation (19)), and set the frequency of this source to $\pm j\sqrt{2}$, the forced response (voltage across a-b) will be zero. (Such a source would be $i(t) = A \cos \sqrt{2}t$. To this current, the LC path looks like a short circuit. Can you verify this?).

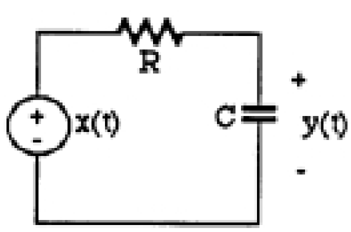
5. Frequency Response Plots

A common way of portraying the behavior of a system is to construct plots of the magnitude of $\mathbf{H}(j\omega)$ and angle of $\mathbf{H}(j\omega)$ vs. ω (frequency response plots). Although these plots directly illustrate only the system's forced response for pure sinusoidal forcing functions, much can be inferred regarding the suitability of the system for use with other waveforms as well. For example, a "flat" gain vs. frequency plot over the audio range for an amplifier is an indication that it will reproduce sound waves (which may not be sinusoidal) in such a way that the ear cannot detect any distortion.

Lowpass Filter:

A lowpass filter is a system which transmits low frequency signals, but attenuates high frequency signals. In the lobby outside a concert hall you may hear only drum and base note sounds because the walls of the building are acting as a lowpass filter, allowing only the low frequency sounds to pass through. A scratchy phonograph record may sound better if a lowpass filter is used to reduce the amplitude of the high frequency scratch-induced sounds while passing the slower music vibrations. The lowpass filter of a car's suspension system blocks sudden bumps the tires may hit from being noticed by the occupants.

The system shown in Figure 12 is a generalized diagram of a very simple simple lowpass filter. The $\mathbf{H}(s)$ transfer function is given on the right. To obtain frequency response plots, replace s by $j\omega$ and form expressions for magnitude and angle of $\mathbf{H}(j\omega)$ vs. ω:



$$\mathrm{H}(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$|\mathrm{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\angle \mathrm{H}(j\omega) = -\tan^{-1}(\omega RC)$$

Figure 12

Lowpass Filter and its transfer function magnitude and angle as functions of ω.

Plots of magnitude and angle of **H**(jω) are given in Figure 13. These plots can be sketched by reference to the expressions in Figure 12 which indicate that for ω=0, the magnitude is 1 and the angle is zero, and that as ω→∞, the magnitude goes to zero and the angle to -90°. One additional point of interest is the frequency at which the magnitude drops to $1/\sqrt{2} = 0.707$ of its highest value. For the LP filter of Figure 12, this occurs at ω = 1/RC, which may be called the "cutoff" or half-power freqency.

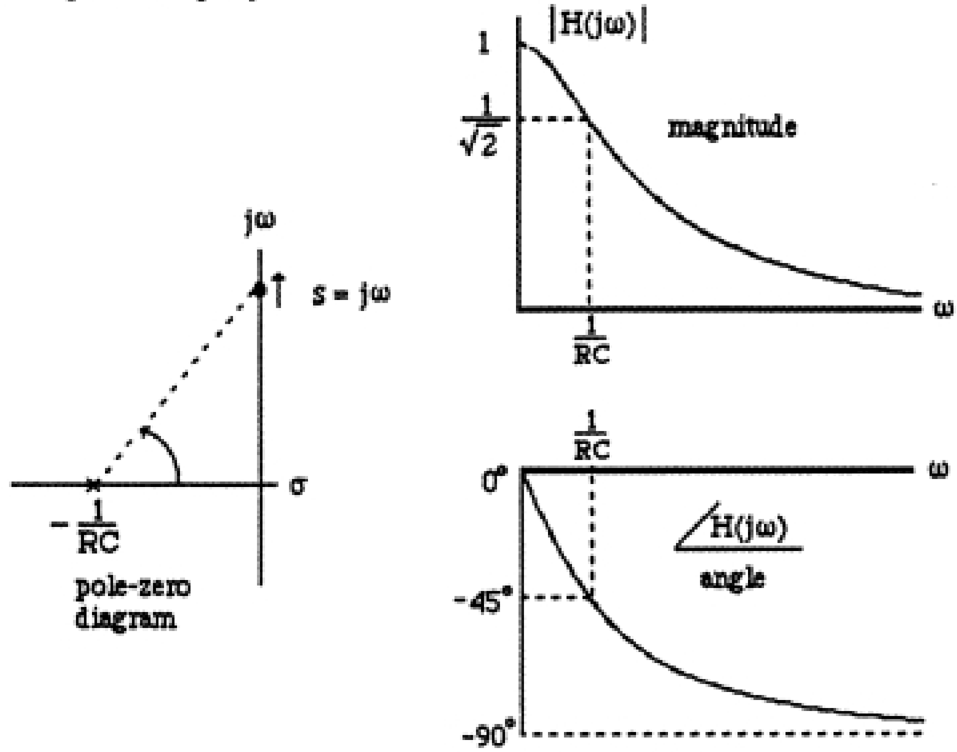


Figure 13

Magnitude and angle of **H**(jω) vs. ω (right), and
pole-zero diagram from which plots can be constructed (left).

The plots of Figure 13 can also be sketched by reference to the system's pole-zero diagram as shown at the left. The difference between two complex numbers can be represented graphically as the distance between them on the complex plane. For example, the complex numbers **A** = 3 - j1, and **B** = 1 + j2, are shown in Figure 14. The difference between these numbers, **A** - **B** = 2 - j3 is represented by the line from **B** to **A**.

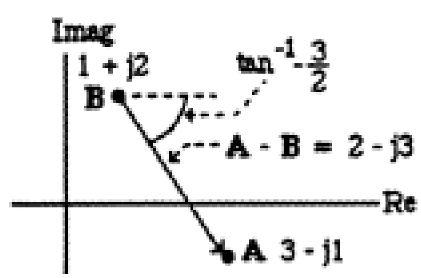


Figure 14
Difference of two complex numbers is the distance between them on the complex

Note that the magnitude of **A** - **B** is $\sqrt{13}$, the length of the line joining the points, and the angle, $\tan^{-1}(-3/2)$, is the angle the line makes with the real axis.

Now consider a typical **H**(s) system function in which the numerator and denominator polynomials have been factored to indicate pole (p_i) and zero (z_i) values.

$$H(s) = K \frac{(s - z_1)(s - z_2) \ldots}{(s - p_1)(s - p_2) \ldots} \tag{20}$$

Each numerator factor and is a difference between two complex numbers (s and a zero of H (s)). Likewise, each denominator factor is the complex difference between sand a pole value' From Figure 14 we saw that such differences can be represented by a line from the particular pole or zero to the point s. To multiply the numerator factors together, we need to multiply their magnitudes (line lengths from zeros), and add their angles (angle with respect to the real axis). To multiply denominator factors, multiply line lengths from poles to point s, and add angles made by these lies with the real axis. Since we need to divide the numerator result by the denominator result (and multiply by scalar K) divide resulting magnitudes and subtract denominator angle from numerator angle. These rules are summarized in Equations 21.

$$\text{magnitude of } \mathbf{H}(s) = K \frac{\text{product of distances from zeros to point s}}{\text{product of distances from poles to point s}}$$

angle of H(s) = (sum of angles from zeros) - (sum of angles from poles) (21)

For example, consider the H(s) function of Equation 18 and its pole-zero pattern, shown in Figure 11. Suppose we want to evaluate H(s) for the sinusoidal frequency s = j 1. If we substitute j 1 for s in Equation 18, each numerator and denominator factor is represented by a line from each zero or pole to the point j1. These lines are shown in Figure 15. We could now determine H(j1) by measuring these line lengths and their angles and applying Equation 21.

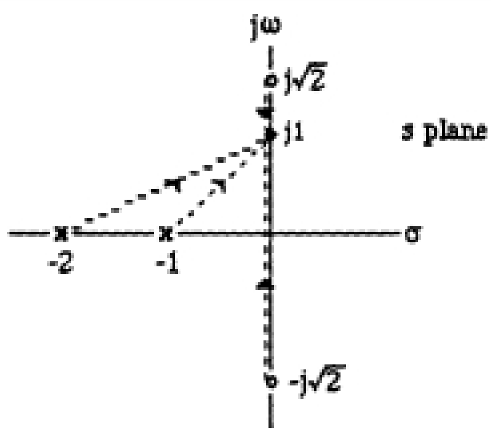


Figure 15
Distances from zeros and poles used to calculate H (j 1)

i

This graphical approach is not particularly useful in the calculation of specific H (s) value: in the manner indicated by Figure 15. It is often used, however, as an aid to the construction of frequency response plots.

Returning to Figure 13, we apply the pole-zero technique to obtain frequency response plots of the lowpass filter. The s value is a point moving up the $j\omega$ axis. The numerator of $\mathbf{H}(s)$ is a constant, 1/RC. The magnitude of the denominator is the distance (shown dotted in Figure 13) from the single pole to the point $s = j\omega$. As the point moves along the $j\omega$ axis, the magnitude vs. frequency plot is generated by dividing the numerator, 1/RC, by the distance from the pole. The angle of $\mathbf{H}(j\omega)$ is the numerator angle (0°), <u>minus</u> the angle from the pole which starts at 0° for $\omega = 0$ and increases to 90° as $\omega \rightarrow \infty$.

Highpass filter:

A highpass filter is one which attenuates low frequency signals, but passes higher frequency signals. A very simple highpass filter is shown in Figure 16.

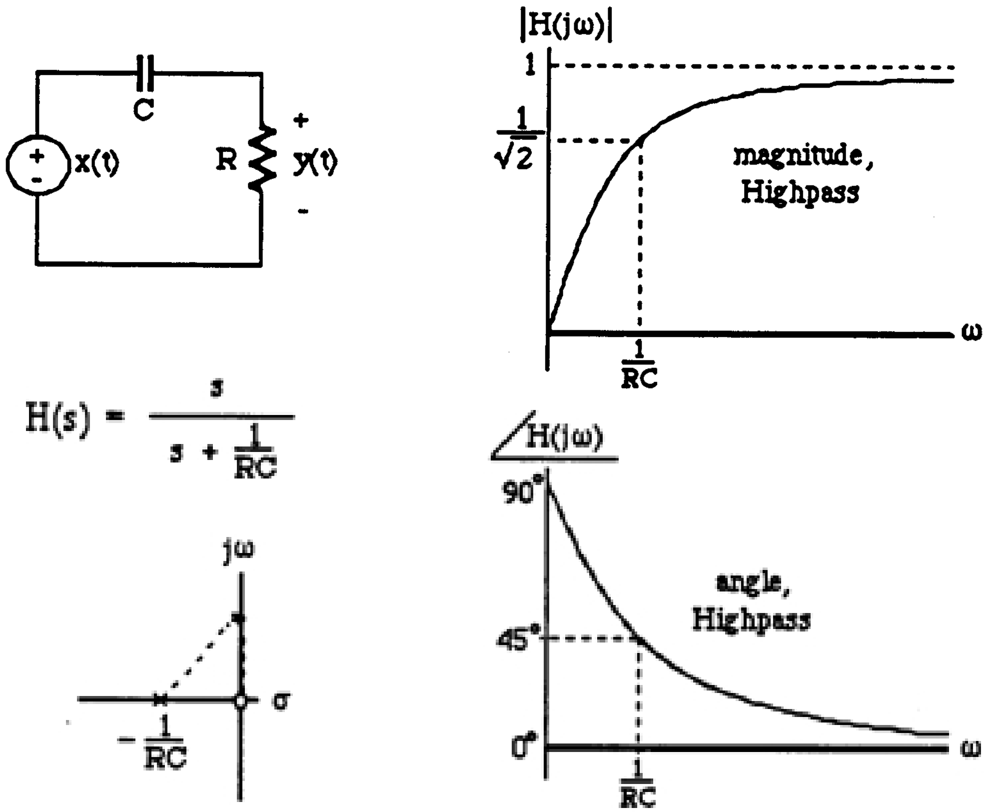


Figure 16
A simple highpass filter with its pole-zero pattern and frequency response plots.

Note that in this case, $\mathbf{H}(s)$ has both a zero and a pole. The distance from the zero is zero at $\omega = 0$, so the magnitude plot starts off at zero. At very low ω, the angle from the zero is 90° and the angle from the pole is ≈ 0. Therefore the angle of $\mathbf{H}(j\omega)$ starts off at (90° - 0°) = 90° As ω increases, the distances from the pole and from the zero both get larger and approach ∞ together, so their ratio approaches unity. Since both angles approach 90°, their difference goes to zero. The results are plotted on the right in Figure 16. Again, the cutoff frequency is $\omega = 1/RC$. The functions are expressed mathematically in Equation 22.

$$|H(j\omega)| = \left|\frac{j\omega RC}{1 + j\omega RC}\right| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\angle H(j\omega) = 90° - \tan^{-1}(\omega RC) \qquad (22)$$

Bandpass Filter:

A bandpass filter is one which passes a narrow range of frequencies (the pass band), but attenuates signals of frequency higher or lower than this range. The simplest form for a bandpass filter is that of Figure 17.

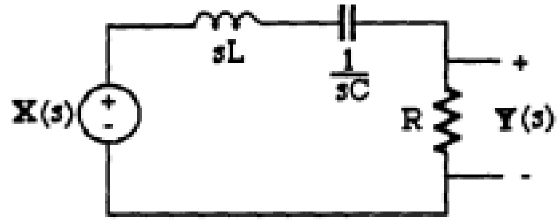


Figure 17
Simple bandpass filter

The transfer function for this system is given by Equation 23.

$$H(s) = \frac{R}{L} \frac{s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad (23)$$

Recall from our work on second order systems that it is convenient to express such quadratic denominators in a more general form:

$$H(s) = K \frac{s}{s^2 + 2\alpha s + \omega_n^2} \qquad (24)$$

where α is called the damping coefficient and ω_n is the undamped natural frequency. A second order system such as represented by Equation 24 operates as a bandpass filter only if it is underdamped. For the underdamped case, the denominator has complex conjugate roots of $-\alpha \pm j\omega_d$ where $\omega_d = \sqrt{(\omega_n^2 - \alpha^2)}$. As the damping coefficient, α, is made smaller compared to the undamped natural frequency, ω_n, the system becomes more frequency selective. The pole-zero pattern given in Figure 18 assumes this condition, showing pole positions close to the $j\omega$ axis ($-\alpha$) compared to their distance from the origin (ω_n). The corresponding frequency response plots given in Figure 18 illustrates the selectivity of this system.

To calculate the frequency of maximum transmission, set $d|H(j\omega)|/d\omega$ equal to zero. The frequency of the maximum is found to be ω_n. Note that this frequncy is not exactly opposite the pole ($j\omega_d$), but slightly farther up the $j\omega$ axis at $j\omega_n$. At $\omega = \omega_n$, the (peak) value of $|H(j\omega_n)|$ is

$K/(2\alpha)$. The bandwidth, B, of the system is usually defined as the frequency difference between half power frequencies (frequencies where the transmission is $1/\sqrt{2}$ of the peak value). We will call these frequencies ω_2 and ω_1. To calculate the half power frequencies, set $|H(j\omega)|$ equal to $K/(2\sqrt{2}\alpha)$ and solve for ω. The exact values for ω_2 and ω_1 are $\sqrt{(\alpha^2 + \omega_n^2)} \pm \alpha$. Note that from this relation, the bandwidth, $B = \omega_2 - \omega_1$, is 2α.

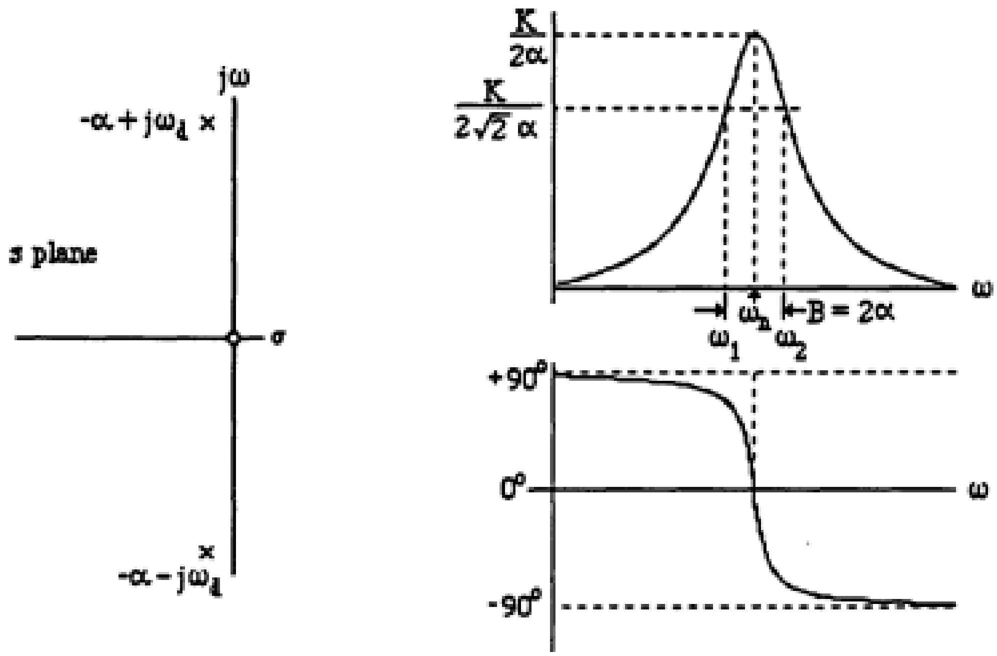


Figure 18
Pole-zero pattern and frequency response plots of the bandpass filter of Figure 17 and Equation 24.

The calculations described above are rather tedious. If $\alpha \ll \omega_n$, certain simplifications can be made to obtain the important quantities more quickly. A convenient factor used to measure the extent of the inequality is Q. Q may be defined as the frequency of maximum transmission divided by the bandwidth, that is, ω_n/B or $\omega_n/(2\alpha)$. When Q is 10 or greater, we may assume $\omega_d \approx \omega_n$, so that the peak occurs at a frequency approximately opposite the pole. The half power frequencies are then symmetrically located on the $j\omega$ axis at points $\pm\alpha$ above and below ω_d. If we assume that the region of the peak is small and far away from the origin and the other pole, the diagram of Figure 19 may be used to sketch the curve.

In Figure 19, since the region of interest near the pole is relatively small ($\alpha \ll \omega_n$), we assume that the distances from the zero and the other pole do not vary appreciably as we move from ω_1 to ω_2. Furthermore, the angle measured from the other pole stays very close to 90° throughout this region. Therefore, the magnitude of $H(j\omega)$ in this region can be approximated as K times the distance from the zero (ω_d) divided by the distance from the lower pole ($\approx 2\omega_d$), and divided by the distance from the upper pole (varies between $\sqrt{2}\alpha$ at ω_1 and ω_2 to α at ω_d). This gives the same result for the peak magnitude as the exact method above, $K/(2\alpha)$. At $s = j\omega_d$, the angle of $\mathbf{H}(s)$ is approximately 90° (angle from the zero) minus 90° (angle from the lower pole), or 0°. (The angle from the upper pole is zero.) At ω_1 and ω_2, the angle from the upper pole is -45o and + 45°, while the angles from the other pole and zero remain the same. Therefore, the angle of $\mathbf{H}(j\omega)$ is ±45° at these frequencies.

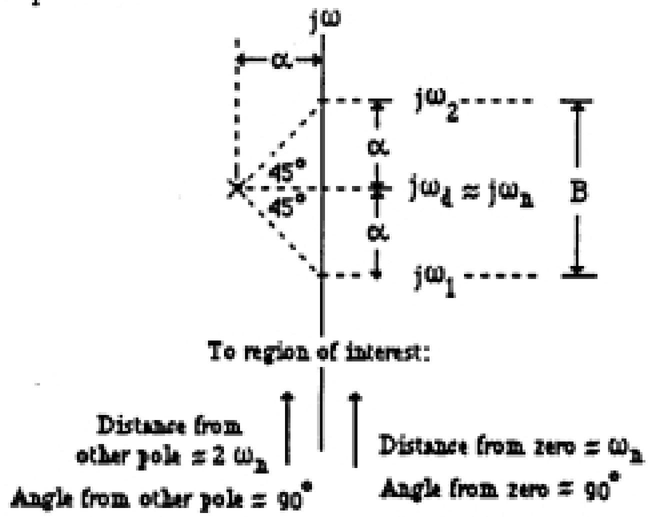


Figure 19

Region of jω axis near the upper pole.

Log-dB or Bode Plots:

The frequency response plots presented in the previous section plotted magnitude vs. frequency on linear scales. In nature, many physical and physiological responses occur on a logarithmic rather than a linear scale. Sound intensity is measured by the average energy received per unit area. However, when sound is received by the ear, its loudness is perceived to be more on a logarithmic scale, that is, the sound seems twice as loud when the intensity increases by an order of magnitude. The bel, a unit named after Alexander Graham Bell is designed to measure energy ratios on this basis. It is simply the loglo of the energy ratio. A more conveniently sized unit is the decibel, abbreviated dB. Its definition is:

$$\text{No. of dB} = 10 \log_{10} \frac{E}{E_0} \tag{25}$$

where E_0 is a reference energy and E is the measured energy. You may have seen references to sound intensities measured in dB, such as normal conversation (60 dB), rock music (110 dB), jet engine (130 dB), etc. These are calculated by using 10^{-12} watts per square meter (the assumed lower limit of human hearing) as E_0.

When energy transport is important in the analysis of systems, the dB unit is often used to express the ratio of output energy per unit time to input energy per unit time. This quantity is called power gain. Within a system, however, we may be more concerned with the amplification of the input across or through variable than with power gain. For example, one stage of an audio power amplifier may simply amplify voltage and have only minute currents associated with its input or output. To apply the dB measure to such cases, we assume that power is in general proportional to voltage squared or current squared. Therefore, we define the dB unit we will use as 10 log10 of the square of the ratio of voltage or current. Because of the log operation, the exponent 2 can be brought out to multiply the 10

$$\text{No. of dB} = 10 \log_{10} \left| \frac{V_2}{V_1} \right|^2 = 20 \log_{10} \left| \frac{V_2}{V_1} \right| \tag{26}$$

A similar expression can be formulated for the ratio of the through variable.

The number of dB represented by a given ratio can often be estimated without actually computing the logarithms. This is possible by realizing that if V2 is a power of 10 times VI, the dB value is simply 20 times that number. For example if the ratio of V2 to V 1 is 1000, this represents 3(20)0 dB. A ratio of 1 is 0 dB. If V2 is smaller than VI, the dB value is negative.
Since the 1og10(2) = .30103, a convenient approximation to add to our table of dB values is that a 2 to 1 ratio is approximately 6 dB (more accurately, 6.02). Table 1 lists some output to input across or through variable ratios or "gains", and the corresponding (in some cases approximate) dB value.

Ratio	dB	Ratio	dB
1	0	$\frac{1}{10}$	-20
$\sqrt{2}$	3	$\frac{1}{2}$	-6
2	6	$\frac{1}{\sqrt{2}}$	-3
10	20		

Table 1
Approximate dB value for several ratios

Because of the log function, the product of two or more ratios results in the addition of their dB values (including negative dB values). For example, if the output voltage of amplifier 1 (voltage gain =10) is fed to the input of amplifier 2 (voltage gain = 4), the overall gain is the product of the individual gains or 40 times. The gain in dB may be found by breaking down the overall gain multiplier into components found in Table 1. 40 = 10 x 2 x 2, so from Table 1 we have 20 + 6 + 6 = 32 dB. If this output now goes to a voltage divider which attenuates the voltage by a factor of 1/8, then -18 dB (3x -6) must be added reducing the overall gain to 14 dB. Note that the gain now is 5. We could find the number of dB representing a gain of 5 by finding any combination of items from Table 1 which multiply together to give 5, for example l Ox 1l2. Then 20dB - 6 dB = 14 dB.

A logarithmic scale is also often used to portray the frequency variable. Frequency response plots are often drawn on semi=log paper with frequency plotted on the log scale and gain plotted in units of dB. These plots and their use in automatic controls was originated by H.W. Bode and therefore are often called Bode plots. Bode plots made for transfer functions of linear systems of the type we have been studying, often are a series of straight lines over wide ranges of frequency. Since cascading of such systems results in the addition of their individual dB gains, the result can be obtained graphically by adding straight lines.

We will develop a library of Bode plots for simple transfer functions of one or two poles or zeros. These can then be combined to produce plots of transfer functions of greater complexity.

Bode Plots:

Definition: **Magnitude plot: Plot of $|H(j\omega)|$ in dB vs. ω on a log scale.**

Angle plot: Plot of the angle of $H(j\omega)$ vs. ω on a log scale.

The technique of constructing Bode plots of H(s) is to consider the multiplying constant, the zeros and the poles separately. After sketching the graph produced by each of these factors, the graphs are added together to obtain the overall plot We will investigate the characteristic plot each of these factors produces:

1. $\mathbf{H}(j\omega) = \mathbf{K}$ (a constant).

Number of dB is a constant, $20 \log_{10}(K)$. Angle is always zero. Bode plot of magnitude is:

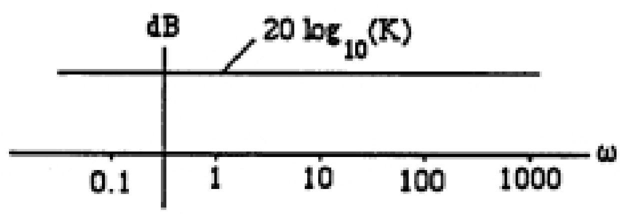


Figure 20

Bode magnitude plot of $\mathbf{H}(j\omega) = \mathbf{K}$. (Angle is zero for all ω)

2. Pole at the origin: $\mathbf{H}(j\omega) = 1/j\omega$.

$dB = 20 \log_{10}(1/\omega) = -20 \log_{10}(\omega)$. The angle is -90^o for all frequencies. Because ω is plotted on a log scale, points such as ω=1, ω=10, ω=100, etc., will be equally spaced. From $-20 \log_{10}(\omega)$, we see that every time ω is increased by a factor of 10, the number of dB reduces by 20. Therefore, the Bode plot is a straight line of slope -20 dB per decade, where decade means a 10 times increase in frequency. Any repeated frequency ratio measures equal steps on the ω axis. For example, ω_1, 2 ω_1, 4 ω_1, 8 ω_1, etc. are also equally spaced frequency points. Each doubling of ω in $-20 \log_{10}(\omega)$ results in an additional - 6 dB. Therefore, the slope of the line could also be described as -6 dB per <u>octave</u>. In addition to the slope, we need one point to plot the line. The easiest value to find is that for $\omega = 1$ where $-20 \log_{10}(1) = 0$. Therefore, the Bode plot is a straight line of slope -20 dB/decade passing through 0 dB at $\omega = 1$:

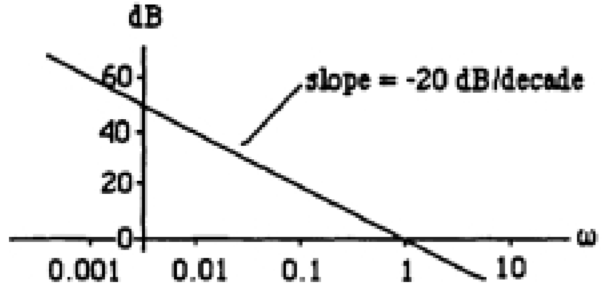


Figure 21
Bode magnitude plot of $\mathbf{H}(j\omega) = 1/j\omega$ (Angle is -90° for all ω).

3. Zero at the origin: $\mathbf{H}(j\omega) = j\omega$.

$dB = 20 \log_{10}(\omega)$. This function has the fixed angle 90°. Its magnitude plot is a straight line of +20 dB per decade, passing through 0 dB at $\omega = 1$.

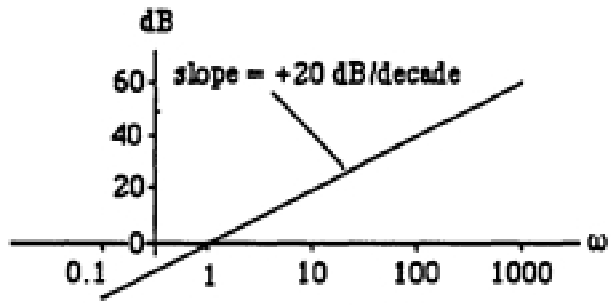


Figure 22
Bode magnitude plot for $\mathbf{H}(j\omega) = j\omega$. (Angle is +90° for all ω.)

Note that for cases 2 and 3, if the function is raised to an integer power, for example, $\mathbf{H}(j\omega) = (j\omega)^{\pm n}$, the slope becomes ±n 20 dB/decade, and the angle becomes ±n 90°.

4. Simple zero in the numerator: $\mathbf{H}(j\omega) = 1 + j\omega/\omega_a$

For this function, consider two frequency ranges: $\omega_a \gg \omega$ and $\omega \gg \omega_a$. For $\omega_a \gg \omega$, $\mathbf{H}(j\omega) \approx 1$, or zero dB with angle 0°. For $\omega \gg \omega_a$, the 1 is negligible. Therefore, $|\mathbf{H}(j\omega)| \approx \omega/\omega_a$. From 3 and 1 above, this is a straight line of slope +20 dB per decade which crosses 0 dB at $\omega = \omega_a$. The angle in the high frequency range is approximately 90°. The straight lines obtained for the two frequency extremes are asymptotes for the true plot. These asymptotes intersect at $\omega = \omega_a$, sometimes called the break frequency or corner frequency. The true curve departs from the asymptotes near the corner frequency. The maximum deviation from the asymptotic value is approximately -3 dB at the corner frequency. Calculations of the actual values, the asmptotic values, and the differences for the frequency range $0.1\,\omega_a > \omega > 10\,\omega_a$ are listed in Table 2.

ω/ω_a	true value dB	asymptote value dB	error dB
0.1	+ 0.04	0	-0.04
0.25	+ 0.26	0	-0.26
0.50	+ 0.97	0	-0.97
1.0	+ 3.01	0	-3.01
1.31	+ 4.35	2.35	-2.0
2.0	+ 6.99	6.02	-0.97
4.0	+12.30	12.04	-0.26
10.0	+20.04	20.00	-0.04

Table 2
Magnitude difference between asymptotes and true curve near break frequency.
Error column gives the error if the asymptotic lines are used in place of the true curve.

Based on the values in Table 2, we may take $0.1\omega_a$ to $10\omega_a$ as the range of frequencies where there is significant departure from the asymptotes for this function. Note that the maximum deviation, 3.01 dB, occurs at the corner frequency. 3.01 dB corresponds to a ratio of $\sqrt{2}$. If the v or f variable decreases by $\sqrt{2}$, the power, proportional to v^2 or f^2, decreases by a factor of 2. Therefore, the corner frequency for this function is sometimes referred to as the half power frequency. The magnitude function is plotted in Figure 23. Outside the range $0.1\omega_a$ to $10\omega_a$, the true curve is very close to the respective asymptotes.

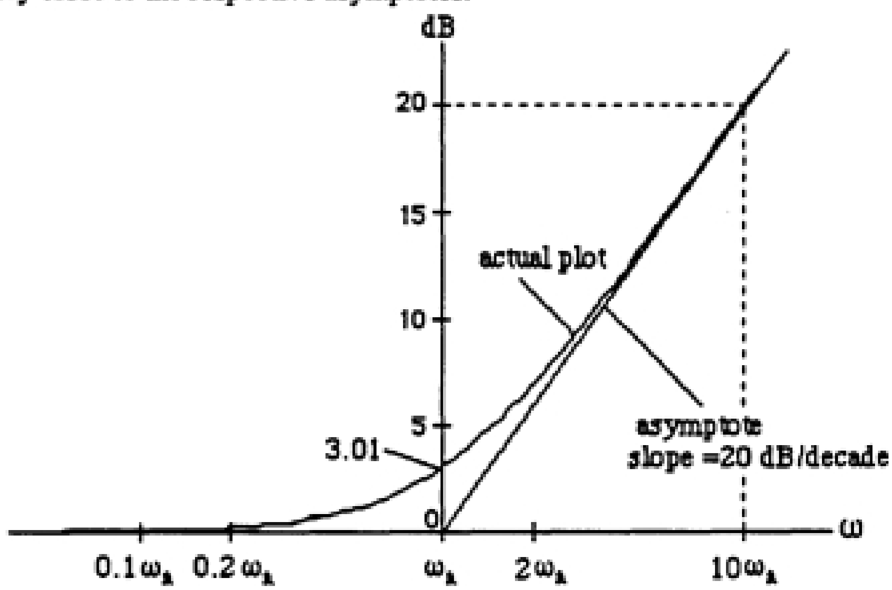


Figure 23
Magnitude plot of $H(j\omega) = 1 + j\omega/\omega_a$

The asymptotic angle of $H(j\omega) = 1 + j\omega/\omega_a$ is 0^o for low ω and and 90^o for high ω. The angle plot is close to these asymptotes outside the region $0.1\omega_a < \omega < 10\omega_a$ (the errors at $0.1\omega_a$ and $10\omega_a$ are 5.7^o. Within this region, a straight line approximation for the frequency function is sometimes used as shown in Figure 24.

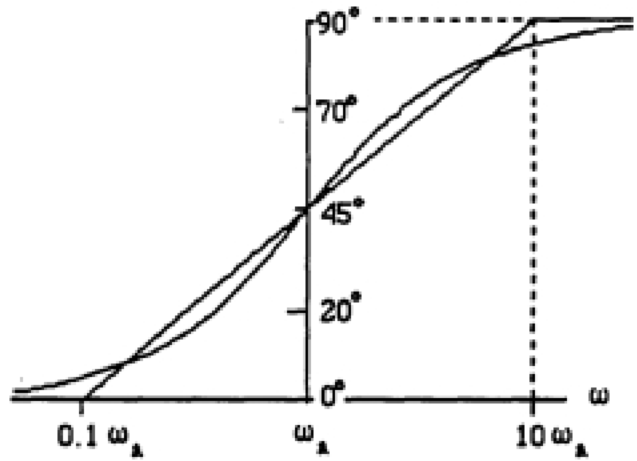


Figure 24
Angle of $\mathbf{H}(j\omega) = 1 + j\omega/\omega_a$ vs ω

5. Simple pole:

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_a}} \quad \text{from which} \quad dB = -20\log_{10}\left|1 + j\frac{\omega}{\omega_a}\right|$$

Taking the log of the fraction yields log of the numerator (0) minus log of the denominator. Except for the negative sign, the result is the same result as in Case 4, above. The low frequency asymptote is again 0 dB for $\omega_a \gg \omega$. For $\omega \gg \omega_a$, the negative sign this time changes the slope to -20 dB per decade. The result is given in Figure 25. The values from Table 2 apply again for this function except that the signs of columns 2, 3, and 4 are reversed.

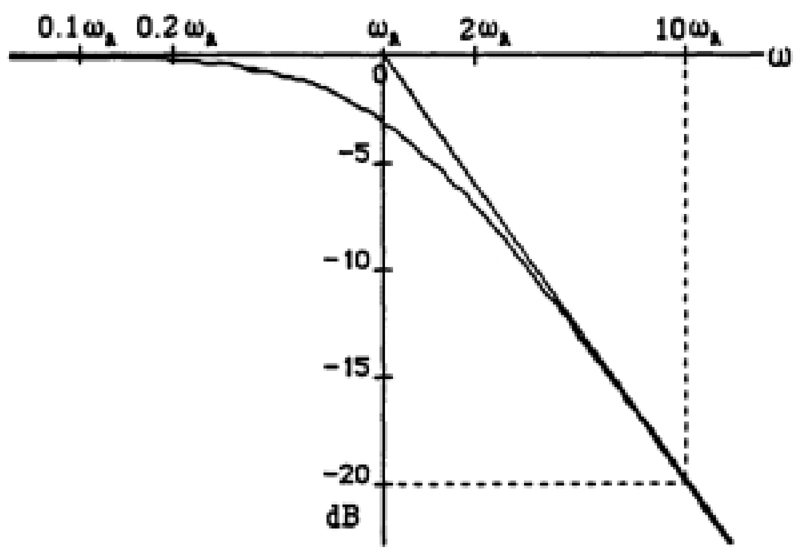


Figure 25
Magnitude plot of $\mathbf{H}(j\omega) = 1/(1 + j\omega/\omega_a)$

The angle asymptotes for this function are zero for $\omega_a \gg \omega$, and now -90° for $\omega \gg \omega_a$. The angle plot is therefore just the (horizontal) reverse of Figure 24. Again, a straight line between $\omega = 0.1\omega_a$, and $\omega = \omega_a$ is often used to approximate this function.

6. Complex pole. Consider $\mathbf{H}(s)$ given below:

$$\mathbf{H}(s) = \frac{1}{s^2 + 2\alpha s + \omega_n^2} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

When the damping ratio, ζ, is <1, complex congugate roots result. Using the second form above for $\mathbf{H}(s)$, set $s = j\omega$ and divide numerator and denominator by $(\omega_n)^2$. The result, after removal of the multiplying constant $(\omega_n)^{-2}$, is:

$$\frac{1}{1 - (\frac{\omega}{\omega_n})^2 + j2\zeta\frac{\omega}{\omega_n}}$$

For $\omega \ll \omega_n$, the magnitude of $\mathbf{H}(j\omega)$ is approximately 1, so the low frequency asymptote is 0 dB. For $\omega \gg \omega_n$, the magnitude is approximately $(\omega_n/\omega)^2$. The second power means that the asymptotic line has a slope of -40 dB/decade (-12 dB/octave). The two asymptotes meet at $\omega = \omega_n$.

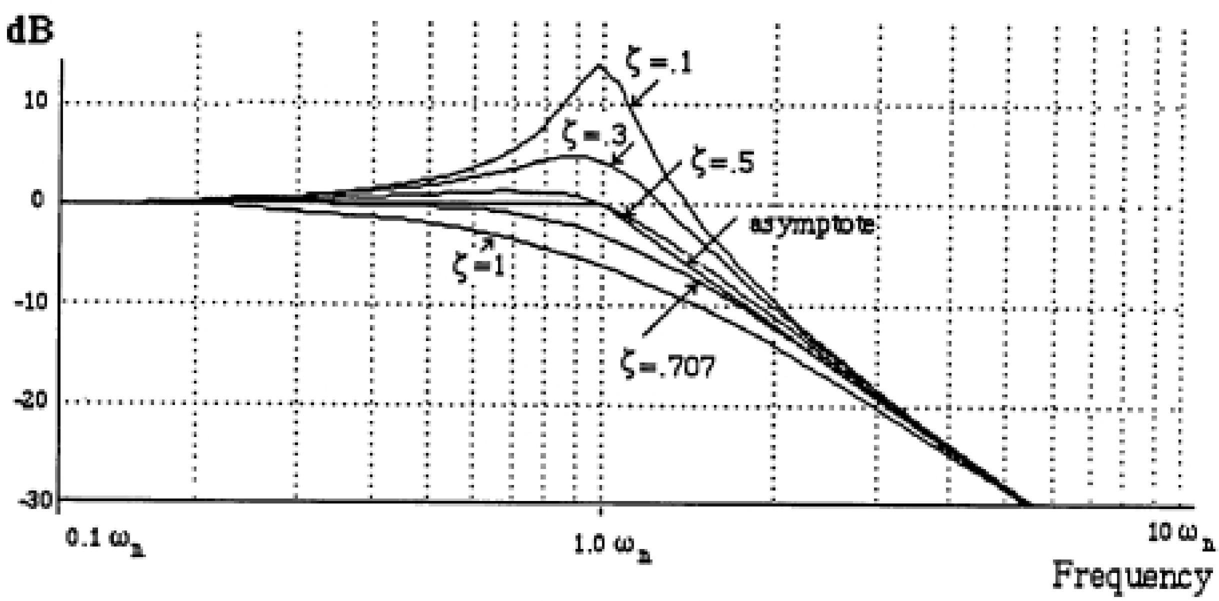


Figure 26
Magnitude plot for quadratic denominator factor.

As the true curve passes the corner frequency, the plot may vary significantly from the asymptotes, depending on the damping ratio as shown in Figure 26. If ζ is small, the plot shows a peak in the vicinity of ω_n. As ζ is increased, the peak diminishes. At $\zeta = 0.5$, the plot passes through the intersection of the asymptotes. For $\zeta = \sqrt{2}$, the plot just stays below both asymptotes. $\zeta = 1$ produces critical damping, a double root at $\omega = \omega_n$. For this case, the error between asymptote and curve at $\omega = \omega_n$ is 6 dB. For $\zeta > 1$, the denominator roots become real and distinct and the result is handled by two sets of asymptotes of the type discussed under #5 above.

The deviation of the true plot of angle of $\mathbf{H}(j\omega)$ for quadratic factors is also a function of ζ. The asymptotes are 0^o for low ω and -180^o for high ω. Again, a straight line may be used to approximate the curve between 0.1 ω_n and 10 ω_n. However, the true angle may deviate significantly from this line as indicated by Figure 27

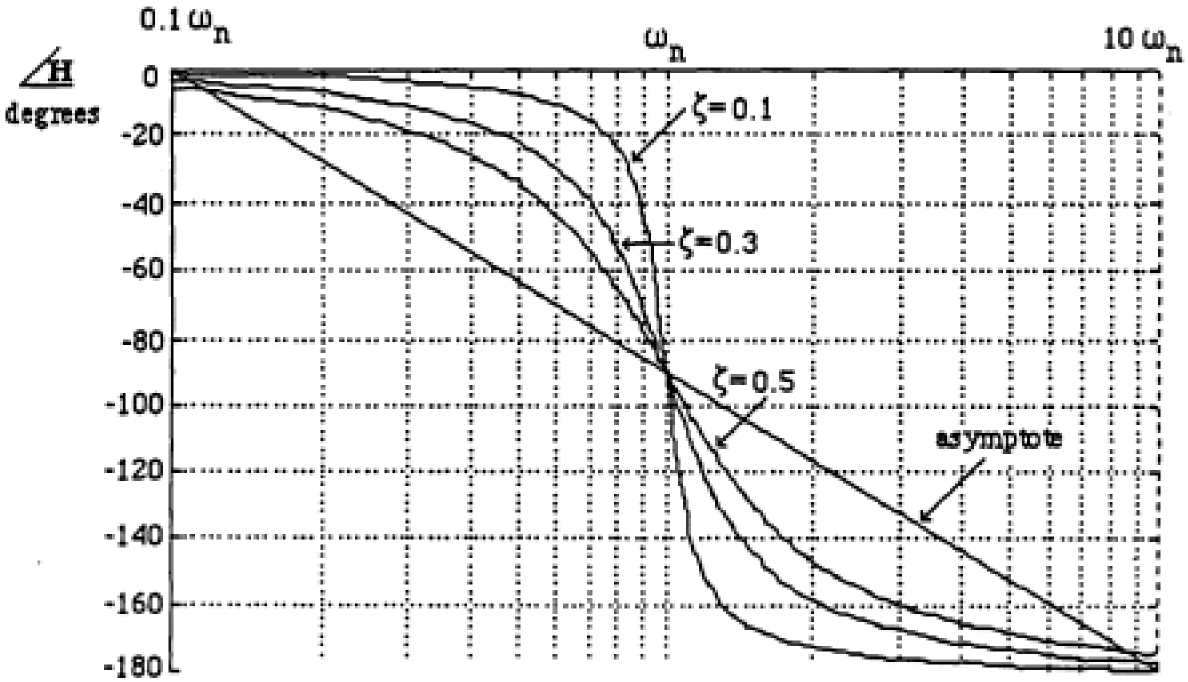


Figure 27
Angle plot for complex conjugate poles.

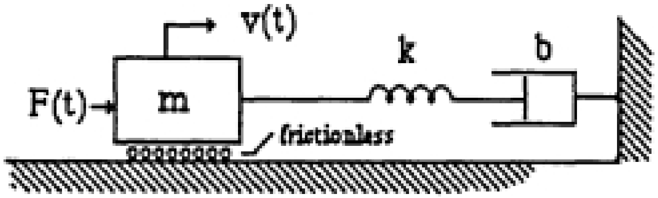
Chapter 12 Problems

1. (a) Find **H**(s), and sketch the pole-zero diagram for a system whose differential equation is

$$\frac{dx}{dt} + 2x = \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 101y$$

(b) If disturbed by a sudden, brief input, what type response would you expect to see from this system?

2. Find **H**(s), the system function, for the system below. The applied force, F(t), is the input, and the resulting velocity, v(t), is the output.
m = 0.5kg, k = 0.5 N/m, b = 0.5 Ns/m.



3. A system has

$$H(s) = \frac{s+2}{s^2+6s+10}$$

(a) Find the system's forced response to the input $x(t) = 4\,e^{-3t}\,u(t)$.
(b) Give the form of the system's natural response.

4. The input, x(t) and output, y(t) of a system are related according to the differential equation given below.
(a) Give the system's transfer function, H(s).
(b) Sketch the pole-zero diagram of H(s). Based on this diagram, what input function of time, x(t), will produce a zero forced response, $y_f(t)$?

$$2\frac{dx}{dt} + x = 2\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 10y$$

5. A system has system function H(s) below. The input is $x(t) = 4\,e^{-2t}\,u(t)$.

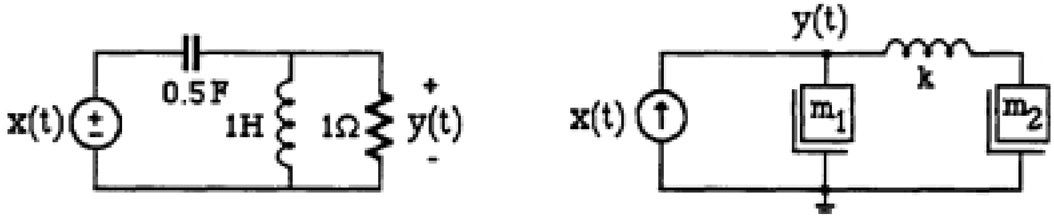
$$H(s) = \frac{s}{s^2+2s+1}$$

(a) Find the forced response, $y_f(t)$.

(b) Find the complete response if $y(0^+) = -8$, and $\frac{dy}{dt}(0^+) = 0$.

6. For each system below,
(a) Find H(s) directly from the diagram
(b) Use H(s) to write the form of the natural response.
(c) Use H(s) to determine the differential equation relating x(t) and y(t).

(d) For the electric system, find the forced response if $x(t) = 4e^{-2t}\,u(t)$.



(b) For k = 0.5, find the forced response if $x(t) = \sqrt{2}\,e^{-t}\cos(t)\,u(t)$.

8. For the system function

$$H(s) = \frac{s^2 + 9}{s(s^2 + 16)}$$

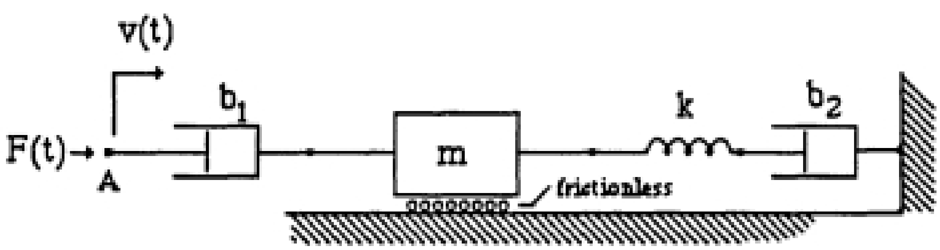
(a) Sketch a pole-zero diagram.
(b) Find the forced response if the input is $5\,e^{-2t}\,u(t)$.
(c) Write the form of the natural response.

9. In the mechanical system shown, a mass slides on a frictionless plane between two damping elements and a spring. A driving force F (the input) is applied at terminal A, which results in velocity v of the terminal (the output). The parameters of the system are:

b1 = b2 = 1 Ns/m   m = 1/Z Ns2/m   k = 2 N/m

(a) Find the system function, H(s). (b) Sketch the pole-zero diagram of H(s). (c) Give the form of the natural part of the velocity response.

(d) Suppose the driver is changed to an ideal velocity source and the response is now defined to be the reaction force of damper b (force at point A). For this case, give the natural form of the reaction force. (e) If the velocity source is v(t) = 12 cos(2t) u(t), find the forced component of the force at point A.



10. Sketch the straight line asymptotes of a log-db (Bode) plot of magnitude and phase angle of H(s). Label all important values, including break frequencies, slope, and angle values.

$$H(s) = \frac{100\,s}{(s + 10)(s + 100)}$$

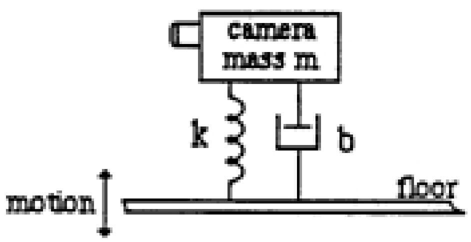
11. For the H(s) function below, sketch a pole-zero diagram. Using this diagram, sketch plots of magnitude and phase angle of H(jω) vs ω on a linear scale.

$$H(s) = \frac{2500\,(s^2 + 64)}{(s^2 + 25)\,(s^2 + 100)}$$

12. Floor vibrations in a TV studio result in a jittery picture. The camera, of mass m, is mounted on a shock absorbing stand which may be modeled as a parallel spring, k, and damper, b, as shown below.

(a) Find H(s), the transfer function between vertical floor velocity V f(s), as an input, and vertical camera velocity, Vc(s), as an output, in terms of m, k, and b.

(b) It is desirable that this system be overdamped (non-oscillatory). If b = 2 and k =1, what range of mass is acceptable for the camera?



Systems Chapter 13 Study, Guide

Laplace Transform

A. Concepts Addressed By This Topic
1. Direct and Inverse Transform.
2. Laplace Transform Theorems
3. Representation of Initial Conditions
4. Complete Solution of System Response

B. Introduction

The Laplace transform is a mathematical operation performed on a function (in our case a function of time) which converts it to a function of a new variable, s. By convention, we use lower case letters to represent time functions *(e.g* , x(t)), and corresponding capital letters to represent their Laplace transforms *(e.g., X(s))*. We will find that s is the same complex variable we have encountered before (s = a + jw). The prime advantage to the application of I-aplace transform to system analysis is that the transform of a derivative or integral of a time function is an algebraic function of s. Therefore, the Laplace transform can be used to convert the differential equations involving a system variable, y(t), to algebraic equations involving its transform, Y(s). After solving the transformed equations for Y(s), we can find the *inverse* transform, which is y(t), the solution of the differential equation.

Laplace transform techniques are most commonly used to determine transient behavior of systems. That is, the dynamics of the system during start-up, or when some sudden change in state occurs such as a sudden application of a load. Often, when systems use feedback loops to aid in control, instability may occur. Pole-zero plots of the system variable, Y(s), can be used to investigate under what conditions instability might result As we have seen already, pole-zero diagrams can also be used to determine many aspects of system behavior.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Determine the Laplace transform of common time functions found in linear systems.
2. Use partial fraction expansion to find the inverse transform of typical Y(s) functions.
3. Solve linear differential equations using Laplace transform.
4. Redraw linear system diagrams with Laplace transformed equivalent elements.
5. Handle initial conditions in energy storage elements using Laplace equivalent circuits.
6. Given a typical system diagram, redraw it in Laplace transformed form, solve for the desired variable as a function of s. then find the inverse transform of that variable.

D. Study Procedure

Read Chapter 13. In addition, almost every referenced book listed in the front of this manual includes a chapter or chapters on Laplace transform. Among the better ones are References 1, 6, 9, and 13.

Chapter 13
Laplace Transform

1. **Definitions:**

The Laplace transform is a mathematical operation which, in our context of system theory, converts a function of time, x(t), into a function of frequency, X(s). The operation is given in Equation 1 and may be referred to as "taking the Laplace transform of x(t)".

$$\mathcal{L}[x(t)] = X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \tag{1}$$

This integral (1) may not converge for all x(t) or for all values of s. For most functions we encounter in systems, the integral will converge over a range of s. For example, if x is a constant, the integral will converge only if $\mathrm{Re}(s) = \sigma > 0$. As long as a value of s can be found which results in convergence, the transform is said to exist. The function X(s) is called the Laplace transform of x(t). Note that function X is not the same function as x. The use of the same letter capitalized is simply a way of pairing the time function with its transform.

Taking the inverse Laplace transform (going from X(s) to x(t)) requires integration in the complex plane. For most applications, we will find easier ways to perform this operation than the use of the complex integral. However, it is given here for completeness.

$$\mathcal{L}^{-1}[X(s)] = x(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds \tag{2}$$

The transform of a function of time and its inverse transform together form a "transform pair". The correspondence of these functions is sometimes noted by the symbol ⇔ as shown below. Remember that this is not an equal sign. It is incorrect to put an equal sign between a time function and its transform!

$$x(t) \Leftrightarrow X(s)$$

We will use the definition to obtain transforms of several common functions: (al x(t) = 1 or u(t) Both have the same transform.)

$$\mathcal{L}[\,u(t)\,] = \int_0^{\infty} (1)\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty} = \frac{\overset{\substack{\text{convergence}\\ \downarrow \text{ for } \sigma > 0.}}{e^{-(\sigma + j\omega)\infty}} - 1}{s} = \frac{1}{s} \tag{3}$$

Note that to make the function vanish at the upper limit, $\mathrm{Re}(s) = \sigma > 0$. Therefore, the region of convergence for this integral is the right half of the s plane.

(b) $x(t) = e^{-at}$ or $x(t) = e^{-at}u(t)$:

$$\int_0^{\infty} e^{-at} e^{-st}\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt = \frac{-1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} = \frac{1}{s+a} \qquad (4)$$

(c) $x(t) = \delta(t)$:

$$X(s) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = 1 \qquad (5)$$

$\uparrow$ choose lower limit to include impulse.

It is useful to construct a table of transforms of the most commonly used functions. Entries in this table can be obtained using the transform definition as was done in Equations 3-5' In addition, **Maple** provides a Laplace transform combed. To determine the Laplace transform of a function of time using Maple, first invoke the library with: readlib(laplace); Then, use laplace(expr ,t,s); where expr is the time function. For example, after the library call, we can find the Laplace transform of x(t) = sin(2t) by typing laplace(sin(2*t),ts);

Certain transform theorems, given below, are helpful in determining transforms. Proof of each theorem is included only where it helps in understanding the use of the theorerm As each theorem is presented, it is used to generate a new entry in the transform table.

2. Transform Theorems

Note: In all theorems below, it is assumed that x(t) and X(s) are a transform

1. Linearity

$$\mathcal{L}[a_1x_1(t) + a_2x_2(t) + \ldots] = a_1X_1(s) + a_2 X_2(s) + \ldots \qquad (6)$$

Example of use:

we know that $\sin(\omega t) = [\frac{1}{2j} e^{j\omega t} - \frac{1}{2j} e^{-j\omega t}]$

From Equation 4, $\mathcal{L}[e^{j\omega t}] = \frac{1}{s - j\omega}$ and $\mathcal{L}[e^{-j\omega t}] = \frac{1}{s + j\omega}$

$$\text{So,} \quad \mathcal{L}[\sin \omega t] = \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{\omega}{s^2 + \omega^2} \qquad (7)$$

2. First Derivative

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = s\,X(s) - x(0^-)$$

Proof: We must evaluate

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt$$

Use integration by parts: $u = e^{-st}$, $dv = dx$:

$$\int_{0^-}^{\infty} u\,dv = u\,v \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\,du = e^{-st} x(t) \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)\,[-s]\,e^{-st}\,dt$$

If the transform exists, $e^{-st}\,x(t)$ goes to zero as $t \to \infty$, so the first term is $-x(0^-)$. After removing $-s$ from the integral, we see that this term is just $sX(s)$, proving Equation 8.

Example of use:

$$\cos \omega t = \frac{1}{\omega} \frac{d}{dt} [\sin \omega t]$$

Therefore, $\mathcal{L}\,[\cos \omega t] = \frac{1}{\omega}\left[s \frac{\omega}{s^2 + \omega^2} - \underset{\substack{\uparrow \\ \sin \omega t \text{ at } t=0}}{0} \right] = \frac{s}{s^2 + \omega^2}$

3. Additional derivatives

We can operate with (8) on the second derivative (derivative of the first derivative) to get:

$$\mathcal{L}\left[\frac{d}{dt}\left(\frac{d}{dt}\, x(t)\right)\right] = s\,[\,sX(s) - x(0^-)\,] - x'(0^-) = s^2 X(s) - s\,x(0^-) - x'(0^-)$$

In general, for the nth derivative,

$$\mathcal{L}\left[\frac{d^n}{dt^n}\, x(t)\right] = s^n X(s) - s^{n-1} x(0^-) - s^{n-2} x'(0^-) - \ldots - \overset{\substack{\text{(primes)} \\ \downarrow}}{x^{(n-1)}}(0^-) \qquad (9)$$

Example of use:

Given: $2\frac{dy}{dt} + 3y = 6\,u(t)$, $\qquad y(0^-) = 0.$

Take transform term by term: $[2sY(s) - y(0^-)] + 3Y(s) = \frac{6}{s}$

$$Y(s) = \frac{6}{s\,(2s+3)} = \frac{3}{s\,(s+\frac{3}{2})} = \frac{2}{s} - \frac{2}{s+\frac{3}{2}}$$

We have already found these transforms (Equations 3 and 4), so the inverse is

$$x(t) = 2[1 - e^{-1.5t}]\,,\ \ t > 0.$$

4. Transform of an integral

$$\mathcal{L}\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right] = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\lambda)\,d\lambda}{s} \qquad (10)$$

In most of our applications, the second term will be zero.

Example of use: Find $\mathcal{L}[\,r(t)]$, the transform of a ramp function.

Now, $r(t) = \int_{-\infty}^{t} u(t)\,dt.$ Since $\mathcal{L}[u(t)\,] = \frac{1}{s}$, then $\mathcal{L}[\,r(t)\,] = \frac{1}{s^2}$

Note that this is easier than evaluating $\int_{0}^{t} t\,e^{-st}\,dt.$

Note also that we now have transforms for the unit ramp, unit step, and unit impulse of $1/s^2$, $1/s$, and 1, respectively. From this we can deduce the transforms of further derivatives or integrals of these singularity functions.

5. Frequency shift

Replacement of s by (s + a) in X(s) is equivalent to multiplication of x(t) by e^{-at}.

$$\mathcal{L}\left[\,e^{-at}\,x(t)\,\right] = X(s + a) \qquad (11)$$

Proof:

$$\int_{0}^{\infty} e^{-at}\,x(t)\ \ e^{-st}\,dt = \int_{0}^{\infty} x(t)\,e^{-(s+a)\,t}\,dt = X(s + a)$$

Example of use: Find $\mathcal{L}[\,e^{-at}\sin\omega t\,]$

We know from Equation 7 that $\sin\omega t \Leftrightarrow \dfrac{\omega}{s^2 + \omega^2}$

So, $\mathcal{L}\left[\,e^{-at}\sin\omega t\,\right] = \dfrac{\omega}{(s + a)^2 + \omega^2}$. Similarly, $\mathcal{L}\left[\,e^{-at}\cos\omega t\,\right] = \dfrac{s + a}{(s + a)^2 + \omega^2}$.

6. Time shift

Shifting a function of time by t_o seconds on the time axis is equivalent to multiplying its transform by e^{-st_o}. This is the dual of the frequency shift theorem just discussed.

$$\mathcal{L}[\, x(t - t_o)\, u(t - t_o)] = e^{-st_o} X(s) \tag{12}$$

Important: To shift the time function, the step function must also reflect the shift!

Example of use: Find the transform of a one half cycle pulse, $x(t) = \sin \pi t$, from zero to 1 on the t axis, zero elsewhere. Solution: The function can be written as the difference of two sinusoids: $x(t) = [\sin \pi t]\, u(t) - [\sin \pi(t-1)]\, u(t-1)$. Then from Equation 7 and the time-shift theorem,

$$X(s) = \frac{\pi}{s^2 + \pi^2} [\, 1 + e^{-s}]$$

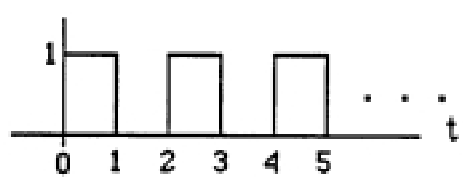
Additional time-shift examples:

Find the transform of the rectangular pulse $x(t) = u(t) - u(t-1)$

This is the transform of a step and the transform of a shifted step:

$$X(s) = \frac{1}{s} [\, 1 + e^{-s}\,]$$

Using this result, find the transform of the waveform below:



The transform of the first pulse is, as we determined above, $X(s) = \frac{1}{s} [\, 1 + e^{-s}\,]$.

If this transform is now multiplied by e^{-2s}, e^{-4s}, etc., we generate the the other pulses in the train (the first pulse delayed by 2, 4, etc.). Therefore, $X(s) = \frac{1}{s} [\, 1 - e^{-s}\,]\, [\, 1 + e^{-2s} + e^{-4s} + \ldots\,]$

The series $1 + e^{-2s} + e^{-4s} + \ldots$ is generated by the fraction $\dfrac{1}{1 - e^{-2s}}$ so

$$X(s) = \frac{1}{s} \; \frac{1 - e^{-s}}{1 - e^{-2s}}$$

From this result we see that it is possible to obtain the transform of a periodic function by finding the transform of one period, then dividing this transform by $1 - e^{-2sT}$, where T is the period.

7. Convolution

The transform of the convolution of two time functions is the product of their individual transforms.

$$y(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\ x_2(t-\lambda)\ d\lambda\ , \quad \text{where}\ x_1(t) \Leftrightarrow X_1(s)\ \text{and}\ x_2(t) \Leftrightarrow X_2(s)$$

If we assume $x1$ does not start before X = 0, the lower limit can be adjusted to zero. Then the transform is:

$$\mathcal{L}\ [y(t)] = \int_0^{\infty} \left(\int_0^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \right) e^{-st}\ dt$$

where the outside integral is on t and the inside integral is on λ. Now reverse the order of integration:

$$\mathcal{L}\ [y(t)] = \int_0^{\infty} x_1(\lambda) \left(\int_0^{\infty} x_2(t-\lambda)\ e^{-st}\ dt \right) d\lambda$$

The interior integral is the transform of a time-shifted function so from (12) it represents $e^{-s\lambda}\ X_2(s)$. Replacing the integral with this expression yields

$$X_2(s) \int_0^{\infty} x_1(\lambda)\, e^{-s\lambda}\, d\lambda \ = \ X_1(s)\ X_2(s)$$

Therefore, $\mathcal{L}[x_1(t) * x_2(t)] \ = \ X_1(s)\ X_2(s)$ (13)

This theorem is very useful in system analysis. We know that the convolution of the input function, x(t), with the impulse response, h(t), yields the output function, y(t). We now see that if we take the Laplace transforms of x(t) and y(t) first, we can multiply these functions of s together to get Y(s), the transform of the output function.

Example of use: A system has an impulse response $h(t) = 2\,e^{-3t}\,u(t)$ and $x(t) = 3\,e^{-4t}\,u(t)$ is applied to its input. Find the resulting output function, y(t).

The transforms are, using Equation 4, $H(s) = 2/(s+3)$ and $X(s) = 3/(s+4)$. Y(s) is the product of these transforms:

$$Y(s) = \frac{6}{(s+3)(s+4)} = \frac{6}{(s+3)} - \frac{6}{(s+4)} \Leftrightarrow (\,6\,e^{-3t} - 6\,e^{-4t}\,)\ u(t) = y(t)$$

The expansion of the original fraction into the two partial fractions enabled us to use Equation 4 to get the inverse transform. The methods of partial fraction expansion will be discussed in the next section.

8. Product of time functions:

Just as the frequency shift and time shift theorems were duals, this theorem is the dual of Theorem 7. The transform of the product of two time functions may be obtained by the complex convolution of their transforms. Unfortunately, because Laplace transforms are complex functions, the convolution integral involves integration over a path in the complex plane. Evaluation of this kind of integral is beyond the scone of this course.

$$\mathcal{L}[\, x_1(t)\, x_2(t)\,] = \frac{1}{2\pi j} \int_{c - j\infty}^{c + j\infty} X_1(s - \lambda)\, X_2(\lambda)\, d\lambda \tag{14}$$

9. scaling

If the variable t is multiplied by a constant in the time function, the transform has its amplitude and the frequency variable s divided by that same constant. As an example, suppose a tape recording of a sinusoidal tone is played at twice normal speed. Then its plot on the time axis will be compressed (be over twice as fast), but its frequency will be expanded in the s domain (pitch an octave higher)'

$$\mathcal{L}[x(at)] = \frac{1}{a}\, X\!\left(\frac{s}{a}\right), \qquad a>0 \tag{15}$$

Proof:

$$\int_0^\infty x(at)\, e^{-st}\, dt = \int_0^\infty x(at)\; e^{-\left(\frac{s}{a}\right)(at)}\, \frac{d(at)}{a}$$

The integral on the right, after bringing the 1/a factor out, is the Laplace transform definition with t replaced by at and s replaced by s/a. Therefore, Equation 15 follows.

10. Multiplication by t:

$$\mathcal{L}[\, t\, x(t)\,] = -\frac{d}{ds}\, X(s) \tag{16}$$

This relation is obtained simply by differentiating both sides of the transform definition (Equation 1) with respect to s.

Example of use: Find the transform of $t\,e^{-at}$. We know the transform of e^{-at} is $1/(s + a)$.

$$-\frac{d}{ds}\left(\frac{1}{s+a}\right) = -\frac{1}{(s+a)^2}$$

This theorem can be used repeatedly to handle multiplication by t^2, t^3, etc.

11. Initial Value Theorem

This theorem permits the determination of the initial ($t=0^+$) value of the time function directly from its transform without first finding the inverse transform.

$$\lim_{s\to\infty} sX(s) = x(0^+) \tag{17}$$

Proof: We start with the transform of the derivative (Equation 8):

$$\mathcal{L}[\frac{dx}{dt}] = \int_0^\infty \frac{dx}{dt} e^{-st}\, dt = sX(s) - x(0^-)$$

Take the limit of both sides as $s \to \infty$. If $\sigma > 0$, $e^{-st} \to 0$ as $s \to \infty$ except for $t = 0$ where e^{-st} is 1. If dx/dt is finite at $t = 0$ (meaning that x(t) is continuous at $t = 0$), then the integral is also zero over this point ($t = 0^-$ to $t = 0^+$). With the integral zero at all points, the equation above becomes $0 = sX(s) - x(0^-)$. Since x(t) is continuous at $t = 0$, $x(0^+) = x(0^-)$ and Equation 17 results.

Suppose x(t) has a step discontinuity at $t = 0$, jumping from $x(0^-)$ to $x(0^+)$. Then the derivative at $t = 0$ is $[x(0^+) - x(0^-)]\,\delta(t)$. The integral is still zero for all t except $t = 0$. At this point the impulse function in the integrand produces the value $x(0^+) - x(0^-)$.

$$\lim_{s\to\infty} \int_0^\infty \frac{dx}{dt} e^{-st}\, dt = \int_{0^-}^{0^+} [x(0^+) - x(0^-)]\,\delta(t)\, dt = x(0^+) - x(0^-) = \lim_{s\to\infty} [sX(s) - x(0^-)]$$

The $x(0^-)$ on both sides of the last equality cancel, again resulting in Equation 17.

Example of use: The output of a system is given below. Find $x(0^+)$.

$$Y(s) = \frac{2s + 3}{s^2 + s + 1}$$

$$\lim_{s\to\infty} s \frac{2s + 3}{s^2 + s + 1} = 2 = x(0^+)$$

12. Final Value Theorem

This theorem permits the determination of the final ($t \to \infty$) value of the time function directly from its transform without first finding the inverse transform.

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{18}$$

Warning: This right side of Equation 18 gives a value whether the left side limit exists or not! If the time function limit does exist, the value obtained from the right side will be correct.

The proof of this theorem is similar to that used for the initial value theorem. Start again with Equation 8, but this time take the limit of both sides as $s \to 0$.

Examples of use:

The first example shows a correct calculation of the final value. The final value of the function $1 - e^{-2t}$ is 1 which is determined below from its transform, using the final value theorem.

$$1 - e^{-2t} \Leftrightarrow \frac{2}{s(s+2)} \qquad \lim_{s\to 0} \frac{s\, 2}{s(s+2)} = 1$$

The second example shows an incorrect result which can occur if the time function does not approach a final value as $t \to \infty$. We try to find the final value of sin ωt. The theorem incorrectly gives an answer of 0.

$$\sin \omega t \Leftrightarrow \frac{\omega}{s^2 + \omega^2} \qquad \lim_{s \to 0} \frac{s\,\omega}{s^2 + \omega^2} = 0$$

The transforms derived so far are summarized in the following table' Additional transforms may be obtained by application of the preceding theorems to the transforms in the table.

Table of Transforms

	x(t)	X(s)		x(t)	X(s)
1.	$\delta(t)$	1	6.	$t\,e^{-at}$	$\frac{1}{(s+a)^2}$
2.	$u(t)$	$\frac{1}{s}$	7.	$e^{-at} \sin bt$	$\frac{b}{(s+a)^2 + b^2}$
3.	e^{-at}	$\frac{1}{s+a}$	8.	$e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2 + b^2}$
4.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$			
5.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$			

4. Inverse Transforms: Partial Fractions

When a system containing ideal sources and R, L, and C ideal elements is solved by writing loop or node equations, the resulting differential equation takes the form

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m x}{dt^m} + b_1 \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_m x$$

If the Laplace transform is taken term-by-term, the left side becomes:

$[a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n]\, Y(s) + [\text{initial condition terms}]$

The initial condition terms form a polynomial in s, as does the coefficient of Y(s).

The right side of the equation transforms into a similar

$$[b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m] X(s) + \text{[initial condition terms]}$$

When this equation is solved for the output transform, Y(s), the result is a fraction of two polynomials in s:

$$Y(s) = \frac{P(s)}{Q(s)} \qquad (19)$$

where P(s) and Q(s) are polynomials with real coefficients. The final step in the solution of y(t) is now to find the inverse transform of Y(s). There are several ways to obtain the inverse transform' The least likely to be practical is to apply the inverse transform integral, Equation 2. A second way is to look up Y(s) in a large tables of transforms which you can buy or find in the library. This is often inconvenient and time-consuming. **Maple** provides an inverse transform function *invlaplace* which is a third way to find inverse transforms. Before using invlaplace, you must first invoke the Laplace library by the command

```
readlib(laplace);
```

Then, for example, if you want the inverse transform of (s + 1)/ s(s + 2), you would use

```
invlaplace((s+l)/s*(s+2),s,t);
```

If neither the large table of transforms nor a Maple equipped computer is available, you must fall back on a technique which can simplify Y(s) into a series of forms whose inverse transforms you already know' The procedure is called "partial fraction expansion". This process forms a separate fraction for each denominator factor of Equation 19. The inverse transform is then taken term by term.

Steps in Partial Fraction Expansion of P(s)/Q(s):

1. Be sure that the numerator polynomial, P(s), is of lower order than the denominator polynomial, Q(s). If this is not the case, perform long division to obtain terms of powers of s and a remainder with numerator of lower order than the denominator:

2. Factor the denominator. Roots will be:

A. Real, distinct. The partial fractions take the form

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{(s - s_1)(s - s_2)\ldots(s - s_n)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \ldots + \frac{K_n}{s - s_n}$$

B. Complex conjugate pairs. Partial fractions are complex conjugates:

$$\frac{s + c}{s^2 + a s + b} = \frac{K_1}{s + \alpha + j\beta} + \frac{K_1^*}{s + \alpha - j\beta}$$

C. Repeated Form as many partial fractions as the power of the factor.

$$\frac{P(s)}{(s+a)^q} = \frac{K_{11}}{(s+a)} + \frac{K_{12}}{(s+a)^2} + \ . \ . \ . + \frac{K_{1q}}{(s+a)^q}$$

3' For each case in step 2, the K values must be determined. This is simply an algebra problem. One way to proceed is to multiply through by the denominator, Q(s), then equate like powers of s. A method which is usually faster is outlined below in the examples'

Example 1: (Numerator not lower order than denominator)

$$Y(s) = \frac{s^2 + 2s + 2}{s + 1}$$

This fraction has numerator of higher order than the denominator. Therefore, use long division:

$$\begin{array}{r|l} & \quad\ \ s + 1 \\ \hline s+1 & s^2 + 2s + 2 \\ & \underline{s^2 + \ \ s \qquad} \\ & \qquad s + 2 \\ & \qquad \underline{s + 1} \\ & \qquad\quad\ 1 \end{array}$$

so $Y(s) = s + 1 + \dfrac{1}{s+1}$

From which $y(t) = \delta'(t) + \delta(t) + e^{-t}\,u(t)$

Example 2A: (Roots real, distinct) $Y(s) = (2s + 3)/ (s^2 + 3s + 2)$

$$\frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

Multiply through by s + 1, then take the limit of both sides as $s \to -1$ (the root):

$$\lim_{s\to -1}\left((s+1)\frac{2s+3}{(s+1)(s+2)}\right) = \lim_{s\to -1}\left|(s+1)\frac{K_1}{s+1}\right| + \lim_{s\to -1}\left|(s+1)\frac{K_2}{s+2}\right|$$

The limits are: $\qquad 1 \qquad = \qquad K_1 \qquad + \qquad 0$

This technique will always result in one numerator constant (of the root used in the multiplication) alone on the right side' We can summarize the process with the

$$K_i = (s - s_i)\frac{P(s)}{Q(s)}\Bigg|_{s \to s_i} \tag{19}$$

Using the same procedure on the second root,

$$K_2 = (s+2)\frac{2s+3}{(s+1)(s+2)}\Bigg|_{s \to -2} = \frac{-1}{-1} = 1$$

$$\text{So, } Y(s) = \frac{1}{s+1} + \frac{1}{s+2} \quad \text{or,} \quad y(t) = e^{-t} + e^{-2t}, \quad t > 0$$

Example 2B: Complex conjugate roots:

$$Y(s) = \frac{s^2 + 6s + 5}{s(s^2 + 4s + 5)} \quad \text{(quadratic formula gives denominator roots of } -2 \pm j\text{)}$$

$$\frac{s^2 + 6s + 5}{s(s+2+j)(s+2-j)} = \frac{K_1}{s} + \frac{K_2}{s+2+j} + \frac{K_2^*}{s+2-j}$$

$$K_1 = \frac{s^2 + 6s + 5}{s^2 + 4s + 5}\Bigg|_{s \to 0} = 1 \qquad K_2 = \frac{s^2 + 6s + 5}{s(s+2-j)}\Bigg|_{s \to -2-j} = j \qquad \text{so } K_2^* = -j$$

$$\text{Then, } Y(s) = \frac{1}{s} + \frac{j}{s+2+j} + \frac{-j}{s+2-j}$$

At this point, there are two possibilities. We can use Transform 3 from our table on each term, then use the Euler identity to combine terms:

$$x(t) = 1 + je^{(-2-j)t} - je^{(-2+j)t} = 1 + je^{-2t}[e^{-jt} - e^{jt}] = 1 + 2e^{-2t}\sin t, \quad t>0.$$

The second possibility is to combine the two complex fractions above,

$$Y(s) = \frac{1}{s} + \frac{2}{(s+2)^2+1} \Leftrightarrow 1 + 2e^{-2t}\sin t$$

A third approach is to separate the original Y(s) fraction into

$$Y(s) = \frac{1}{s} + \frac{As+B}{s^2+4s+5}$$

The 1 numerator of the first fraction was found as illustrated above. The second fraction numerator must allow for an s term as well as a constant, since the denominator has s2. A and B can be found by performing the subtraction of 1/s from Y(s), or by multiplying through by the denominator of Y(s) and then equating like powers of s. The result is A = 0 and B = 2

Example 2C: Repeated roots:

$$Y(s) = \frac{2s^2+3s+2}{(s+1)^3} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2} + \frac{K_{13}}{(s+1)^3}$$

1. (a) Multiply through by $(s+1)^3$: $\quad 2s^2+3s+2 = K_{11}(s+1)^2 + K_{12}(s+1) + K_{13}$

   (b) let $s \to$ the root, -1 $\quad 2 \quad -3 \quad +2 = \quad 0 \quad + \quad 0 \quad + K_{13}$

   So K13 = 1. But this method won't work for K_{11} (try it).

2. To evaluate K_{11}, differentiate through the equation of 1(a) above with respect to s:

   $4s+3 = 2K_{11}(s+1) + K_{12}.$

   as $s \to -1$, $-4+3 = 0 + K_{12}$ $\quad$ so $K_{12} = -1$

3. To evaluate K_{13}, differentiate again with respect to s:

   $4 = 2K_{11}$, $\quad$ so $K_{11} = 2.$

$$Y(s) = \frac{2}{(s+1)} + \frac{-1}{(s+1)^2} + \frac{1}{(s+1)^3} \Leftrightarrow 2e^{-t} - te^{-t} + t^2e^{-t}, \; t>0.$$

The first two inverse transforms are in our table (#3 and #6); the third can be derived using Theorem 10.

Maple also provides a partial fraction expansion standard library function. The call is convert(f,parfrac,s); where f is the function of s to be expanded. For example, to solve the previous example using Maple, type

```
convert((2*s^2+3*s+2)/(s+l )^3,parfrac,s);
```

5. **Applications of Laplace Transforms to Systems**

In our discussion of forced response, we defined H(s) as the ratio of output to input when the forcing function was an exponential function. We will now expand this concept to show that H(s), as well as impedance and admittance, can be found using Laplace transform, and that the complete response of linear systems to most inputs can be found in this manner.

Assume that for each of our basic elements, R, L, and C, we have an input driver which delivers an across variable, v, and a response which is the resulting through variable, f. We wish to find H(s) for each element (H(s) in this case is impedance, Z(s)).

generalized Resistor. $v = fR$

Take the Laplace transform of both sides of this equation: $V(s) = F(s)\,R$
Then,

$$ZR(s) = R \qquad (20)$$

Generalized Inductor:

$$v = L\frac{df}{dt}$$

'The Uplace transform of this equation is:

$$V(s) = sLF(s) - U(0-) \qquad (21)$$

We cannot divide V(s) by F(s) to get a constant for Z as we did with the resistor.
However, if we specify that f(0') is zero, then the impedance becomes

$$ZL(s) = sL \qquad (22)$$

$$f = C\frac{dv}{dt}$$

The Laplace transform of this equation is:

$$F(s) = sCV(s) - C\, v(0-)$$

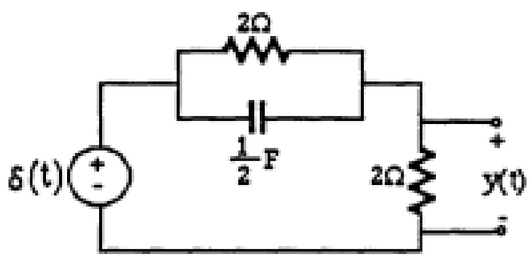
(23)

Again, we cannot divide V(s) by F(s) to get a constant for Z But, if we specify that the initial value of v is zero, then this impedance becomes

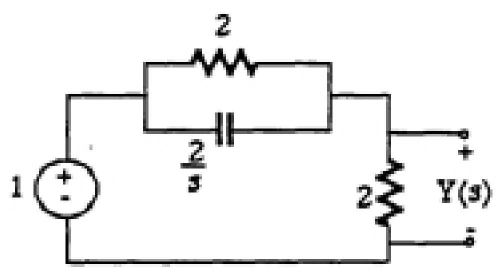
$$Z_C = \frac{1}{sC} \tag{24}$$

These impedance definitions (Equations 22 and 24) enable us to solve for the complete response of systems without first writing the system differential equations. This is illustrated in the examples below. While the restriction of no initial energy storage seems to limit the applicability of this method, we will develop a way to handle initial conditions in the next section. Example

Find the impulse response of the system below:



Solution: Re-draw the system using Laplace transformed impedances and sources. Replace each element R, L, or C value by its Laplace impedance. Replace each source value by its Laplace transform (the transform of $\delta(t)$ is 1):



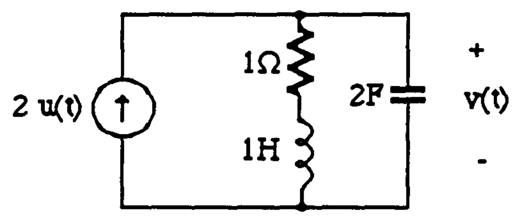
The parallel RC combination has $Z_{RC} = \dfrac{(2)\left(\frac{2}{s}\right)}{2 + \frac{2}{s}} = \dfrac{2}{s+1}$

An expression for the output, Y(s), can now be written using the voltage divider concept:

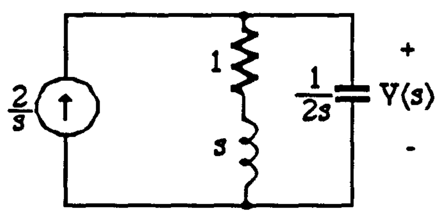
$$Y(s) = (1)\ \frac{2}{2 + Z_{RC}} = \frac{s+1}{s+2}$$ ← Numerator same order as denominator. Must use long division!

$$H(s) = Y(s) = 1 - \frac{1}{s+2} \Leftrightarrow \delta(t) - e^{-2t}\,u(t) = h(t)$$

Example: Find v(t) in the circuit below:



The transform of the step is 1/s, the impedances are $Z_L = s$, $Z_R = 1$, and $Z_C = 1/(2s)$. Redraw using these Laplace values:



The current source works into Z(s) which is the parallel combination of Z_C and Z_{RL}.

$$Z(s) = \frac{(1+s)\left(\frac{1}{2s}\right)}{1 + s + \frac{1}{2s}} = \frac{\frac{1}{2}(s+1)}{s^2 + s + \frac{1}{2}}$$

The voltage is then I(s) times this impedance:

$$V(s) = \frac{2}{s}\,Z(s) = \frac{s+1}{s\left(s^2 + s + \frac{1}{2}\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + s + \frac{1}{2}}$$

We have used a slightly different approach to set up the inverse transform for this function. We know that the quadratic factor can be expanded into two partial fractions. If these two fractions are recombined, they will in general yield a fraction like the second one on the right. Using the usual procedure on A (Equation 19), we get $A = 2$. Now, multiply through by $s(s^2 + s + 1/2)$, to get the equation $s + 1 = 2(s^2 + s + 1/2) + Bs^2 + Cs$. Equating like powers of s on both sides of this equation gives, for the s^2 terms, $0 = 2 + B$, and for the s terms, $1 = 2 + C$. From these equations, we have $A = 2$, $B = -2$, and $C = -1$.

$$V(s) = \frac{2}{s} + \frac{-2s-1}{s^2+s+\frac{1}{2}}$$

The second term can be put into a form found in the table if we complete the square in the denominator.

$$V(s) = \frac{2}{s} + \frac{-2s-1}{s^2+s+\frac{1}{2}} = \frac{2}{s} + \frac{-2s-1}{s^2+s+\frac{1}{4}+\frac{1}{4}} = \frac{2}{s} + \frac{-2\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}$$

The inverse transform is then $v(t) = [\,2 - 2\,e^{-t/2}\cos(t/2)\,]\,u(t)$

Of course, this could have been found directly from the original V(s) using **Maple**:

```
invlaplace((s+1)/(s*(s^2+s+0.5)),s,t);
```

6. Initial Energy Storage

The use of the impedance values defined in Equations 21, 22, and 24 makes possible the formulation of systems problems directly in Laplace transform notation, without the need to first write differential equations. However, these relations were derived under the assumption that all initial energy storage was zero' We will now remove this requirement.

Generalized capacitor with initial energy storage:

An A-type element has stored energy related to the across variable, v. Figure 1 shows a generalized capacitor with across variable Vo at t = 0. v(t) and f(t) are the across and through variables at the terminals at all times.

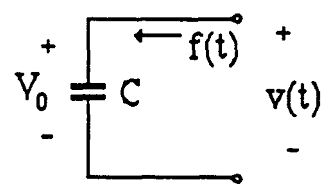


Figure 1
Capacitor initially charged to V_0

The relationship between v and f for this capacitor, good for t > 0, is

$$v(t) = V_0 + \frac{1}{C}\int_0^t f(t)\,dt \qquad (25)$$

We would like to replace this element with its Laplace equivalent. Taking the Laplace transform of equation 25 yields

$$V(s) = V_0/s + (1/sC)(F(s)) \qquad (26)$$

We want to construct a Laplace equivalent circuit which fits Equation 26. Equation 26 looks like the equation of a Thevenin equivalent. The voltage, V(s), is the sum of two (series) voltages. Vo/s is a constant which represents the open circuit voltage of the capacitor (the initial voltage remains if the capacitor is left open-circuited). The other term is the capacitive impedance (1/sC) multiplied by the through variable (F(s)). This is the voltage drop across the "dead" or uncharged capacitor. The circuit shown in the left panel of Figure 2 satisfies equation 26. If we convert this circuit to its time domain equivalent, we get the circuit in the right panel of Figure 2. Either of these equivalent circuits (depending on the domain) may replace a capacitor with initial energy storage in a system diagram.

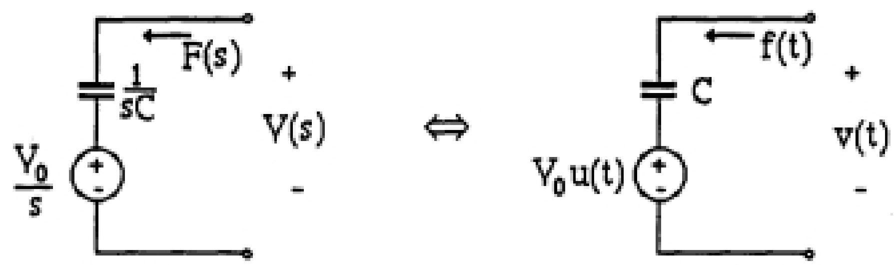


Figure 2
Thevenin equivalent circuit of a generalized capacitor with initial energy storage
Laplace domain (left), time domain (right).

We can replace the circuit on the left of Figure 2 with its Norton equivalent, giving the circuit in the left panel of Figure 3. Either the Thevenin or Norton form may be used as a Laplace replacement for the charged capacitor. For each Laplace circuit, the time domain equivalent is given in the right panel. None of these pictures apply for t < 0.

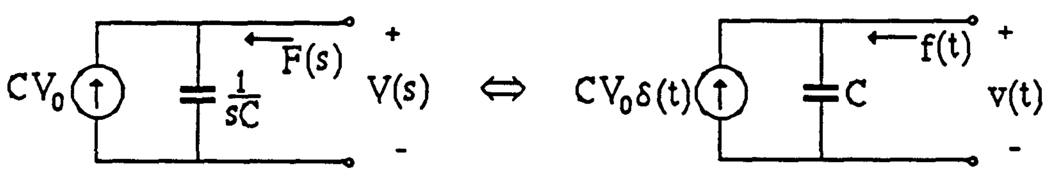


Figure 3
Norton equivalent circuit of a generalized capacitor with initial energy storage
Laplace domain (left), time domain (right).

Generalized inductor with initial energy storage:

A similar analysis provides equivalent circuits for the inductor with initial energy storage: A T -type element has stored energy related to the through variable, £ Figure 4 shows a generalized inductor with through variable f(0) _ - FO at t = 0. v(t) and f(t) are the across and through variables at the terminals at all times.

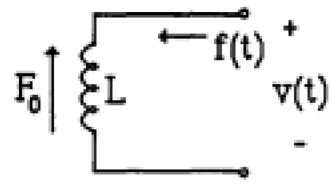


Figure 4
Inductor with initial energy storage. $f(0) = -F_0$

The relationship between v and f for this inductor is

$$f(t) = -F_0 + \frac{1}{L}\int_0^t v(t)\ dt \qquad (27)$$

We would like to replace this element with its Laplace equivalent Taking the Laplace transform of equation 27 yields

$$F(s)\ \_\ - Fo/s + (1/sL)(V(s)) \qquad (28)$$

We want to construct a Laplace equivalent circuit which fits Equation 28. Equation 28 looks like the equation of a Norton equivalenl The through variable , F(s), flowing in at the terminals breaks into two (parallel) parts. -FD/s is a *constant which* represents the short circuit flow through the inductor (the initial flow remains constant if the inductor is short-circuited). The other term is the applied voltage divided by the in The circuit shown in the left panel of Figure 5 satisfies equation 28. If we convert this circuit to its time domain equivalent, we get the circuit in the right panel of Figure 5. Either of these equivalent circuits (depending on the domain) may replace an inductor with initial energy storage in a system diagram.

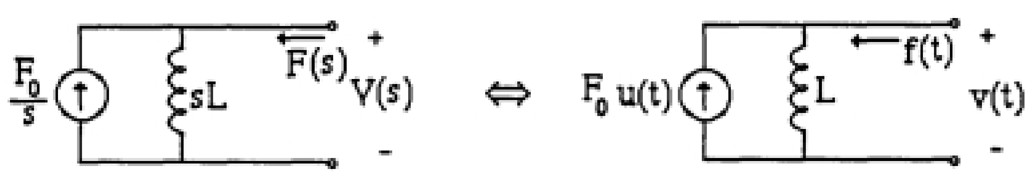


Figure 5
Norton equivalent circuit of a generalized inductor with initial energy storage

We can replace the Laplace circuit on the left of Figure 5 with its Thevenin equivalent The result is the circuit in the left panel of Figure 6. Either the Norton forth (left panel of Figure 5) off. Thevenin form (left panel of Figure 6) may be used as a Laplace transform replacement for the

To get the voltage across the 5Ω resistor, use the voltage divider concept:

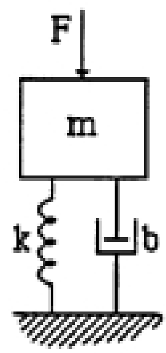
$$V(s) = 2\frac{5}{s+5+\frac{6}{s}} = \frac{10\,s}{s^2 + 5\,s + 6} = \frac{K_1}{(s+3)} + \frac{K_2}{(s+2)}$$

Multiply V(s) by (s + 3) and let s→ -3 to get $K_1 = 30$. Multiply by (s + 2) and let s→ -2 to get $K_2 = -20$. Then

$$v(t) = [30\,e^{-3t} - 20\,e^{-2t}]\,u(t).$$

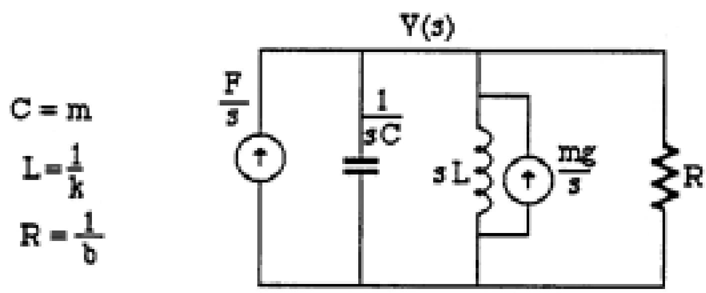
<u>Example:</u>

The platform of a scale has mass m' It is supported by a spring-damper system as shown below. At time t = 0, a weight of 8.2 N is placed on the scale. Determine the velocity of the scale platform for t > 0 if m = 1 kg, k = 9 N/m, and b = 6 Ns/m Use g = 9.8 m/s2.



Solution:

Re-draw the system using Laplace equivalents. The mass has zero velocity at t = 0, so no initial condition equivalent is needed for the capacitor below. The spring does have energy stored at t = 0' because the force due to the weight of the mass (mg) flows to ground through the spring prior to the application of F 'Therefore, the inductor below needs a parallel through source to represent this initial condition.



Write a vertex equation at the top

$$\frac{mg}{s} + \frac{F}{s} = V(s)\ [\, sC + \frac{1}{R} + \frac{1}{sL} \,]$$

$$V(s) = \frac{\frac{9.8}{s} + \frac{8.2}{s}}{s + 6\,s + 9} = \frac{18}{(s+3)^2} \Rightarrow 18\, t\, e^{-3t}\, u(t)$$

Chapter 13 Problems

1. Find the Laplace transform of the following by direct application of the definition integral.

(a) $x(t) = 2\,u(t-2)$ (b) $y(t) = e^{-at} + 3e^{-bt}$ (c) $v(t) = u(t) - u(t-1)$

(d) $x(t) = 4\,e^{-3t}\,u(t-4)$

2. Using only transform theorems and the given transform

$$\pounds(e^{-at}) = \frac{1}{(s+a)}$$

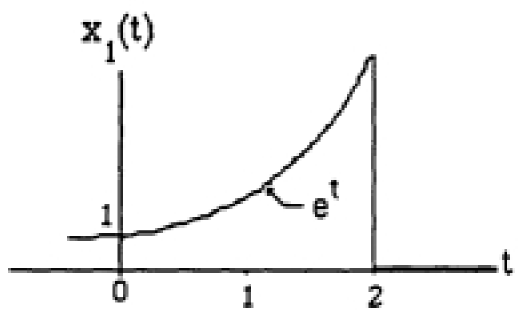
find the Laplace transform of each of the following functions without integrating.

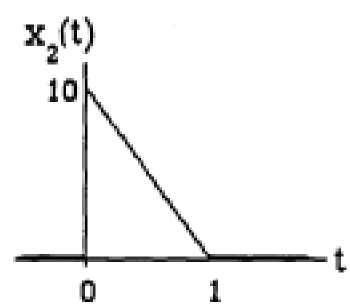
(a) $x(t) = 2e^{-3t} + 3\,e^{-2t}$ (b) $x(t) = 2\,e^{-3(t-4)}\,u(t-4)$.

(c) $y(t) = 4\cos(2t)$ (Hint: use Euler's Identity) (d) $y(t) = t^2\,e^{-3t}$

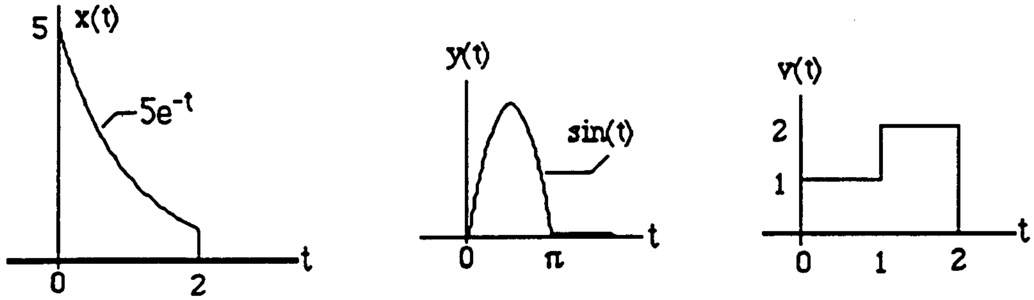
(e) $y(t) = 2 + t$ (e) $x(t) = e^{-2t}\cos(t)$

3. Find the Laplace transform ($X_1(s)$ and $X_2(s)$) of each of the functions pictured below.





4. Find the Laplace transform (X(s), Y(s), and V(s)) of the functions pictured below:



5. Check your answers to all parts of Problems 1-5 using Maple. You can use either the integral operation:

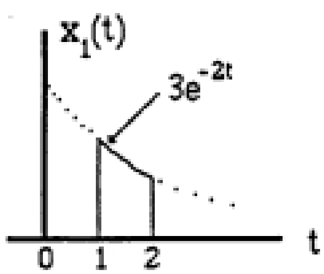
example: to solve 4(a): int(5*exp(-t)*exp(-s*t),t=0..2);
to solve 1(d): int(4*exp(-3*t)*exp(-s*t),t=4'.infinity);
(note: if the upper limit is infinity, first set signum(s) :=l; which guarantees positive s so that Maple finds convergence)

or use the library Laplace function which must be first called by : readlib(laplace); then,

laplace(expr,t,s); where expr is the function of t to be transformed.
e.g. 2(d) laplace(tA2*exp(-2*t),t,s);

6. Find the Laplace transform, X(s), for each x(t) given below:

$x2(t) = t \sin(2t)$



$x_3(t) = \delta(t-2)$

$x_4(t) = 3 + t^2$

7. Find the inverse Laplace transform of each of the following functions:

(a) $F(s) = \dfrac{s^2 - s + 1}{s^2\,(s + 1)}$ (b) $Y(s) = \dfrac{s^2 + 6s + 5}{s\,(s^2 + 2s + 5)}$

(c) $X(s) = \dfrac{s + 1}{s^3 + 4\,s^2 + 4\,s}$ (d) $V(s) = \dfrac{s\,(s + 1)}{s^2 + 2s + 1}$

(e) $X(s) = \dfrac{10\,(s+1)}{s\,(s^2 + 2\,s + 2)}$ (f) $Y(s) = \dfrac{20s}{s^2 + 2\,s + 17}$

8. Find the inverse Laplace transform of each function below:

$$Y(s) = \frac{4(s+1)}{s(s+2)^2} \qquad Y(s) = \frac{2s}{(s+1)(s^2+2s+2)}$$

$$Y(s) = \frac{5\,s^2 + 2\,s + 20}{s^3 + 4\,s}$$

9. Repeat problems 7 and 8 using Maple. The commands for inverse Laplace transform are:

readlib(laplace);   needed once to invoke laplace library.
invlaplace(Y(s),s,t); So 7(e) is invlaplace(10*(s+l)/S*(SA2+2*s+2),s,t);

10' Solve the following differential equations using Laplace transform:

(a) $\dfrac{d^3y}{dt^3} + y = 4\,\delta(t),$ $y(0) = 2,\ y'(0) = 0,\ y''(0) = 2$

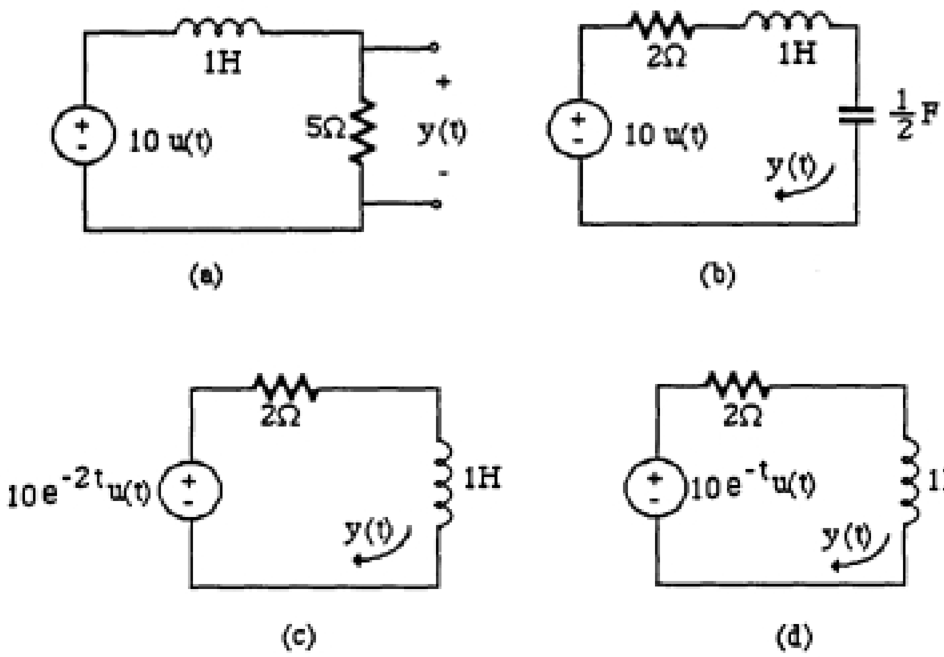
(b) $\dfrac{d^3y}{dt^3} + 14\dfrac{d^2y}{dt^2} + 44\dfrac{dy}{dt} + 40\,y = 10\,u(t),$ $y''(0) = -26,\ y'(0) = 4,\ y(0) = 1/2$

(c) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4\,y = u(t),$ $y(0) = 0,$ $y'(0) = 1$

Answer (c): $[1/4][1 - e^{-2t} + 2\,t\,e^{-2t}]\,u(t)$

11. Repeat problem 10 using Maple and the dsolve function.

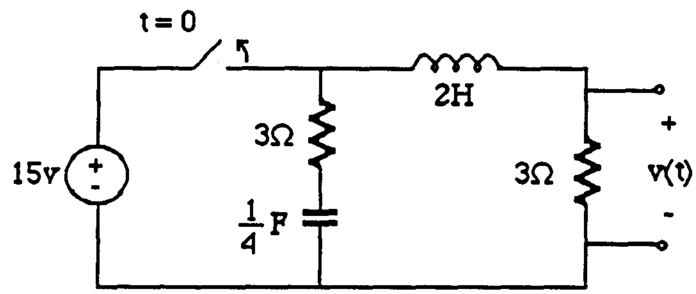
12' For each system below,
(a) redraw the system using Laplace transform elements.
(b) solve for Y(s)
(c) convert to y(t).



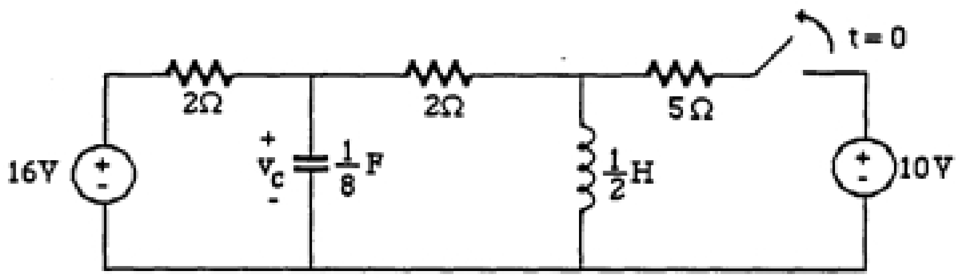
13. A mass is pushed by steady force of 10N. It slides over an oil film with viscous friction of b=2N ' s/m. Suddenly (at t = 0), it moves onto a region where b = 8. Find its velocity as a function of time.

14. The impulse response of a certain system is $h(t) = e^{-t}\,u(t)$. Determine its response to an input $x(t) = 10\,e^{-2t}\,u(t)$ by application of Laplace transform.

15. The switch was closed for a long time, then opened at t = 0.
(a) Draw the Laplace transform circuit including initial conditions.
(b) Obtain V(s), the Laplace transform of the output voltage.
(c) Find the inverse transform, v(t). (Ans. $2\,e^{-t} - e^{-2t}$, $t > 0$.)

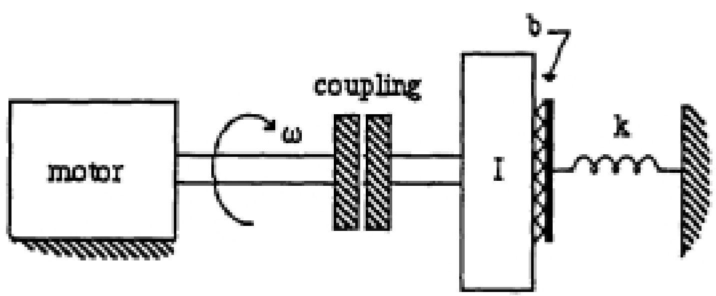


16. The circuit shown was in operation for a long time when, at t = 0, the switch opened. Find $v_c(t)$ for $t > 0$. (Ans.[$8 - 4\,e^{-4t} \sin 4t$] , $t > 0$)

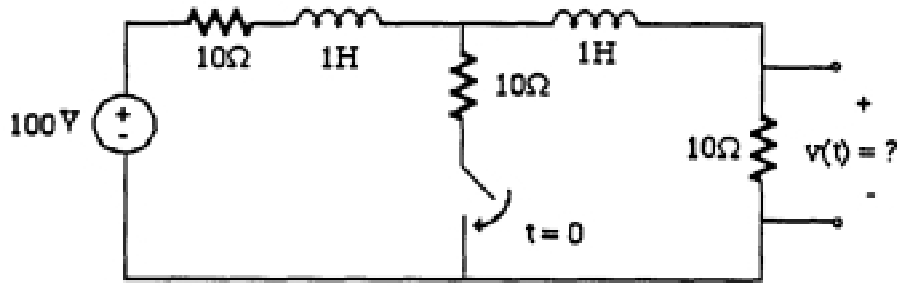


(c) (d)

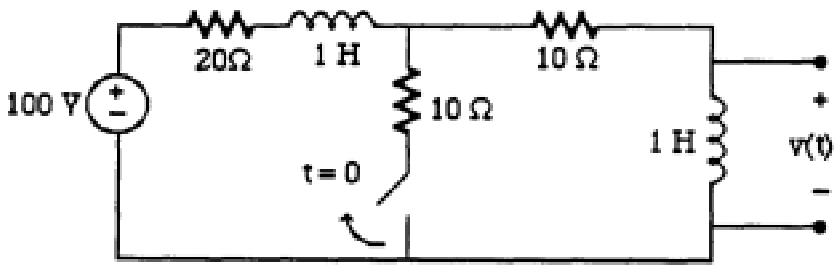
17. The motor on the left drives a rotating flywheel (moment of inertia I = 1/4 Nms^2/r) at a constant angular velocity of ω = 8 r/s through a mechanical coupling. The rotating flywheel makes frictional contact (b = 1/4 Nms/r) with a plate of negligible mass, connected to a rotational spring (k = 1/2 Nm/r). The coupling suddenly disconnects at t = 0. Find the angular velocity of the flywheel for t > 0.



18. The circuit was in operation for a long time when, at t = 0, the switch closes. Find the voltage across the 10Ω resistor on the right for t > 0. (Ans. (1/3)[$100 + 50\,e^{-30t}$] u(t)



19. The circuit shown below is in equilibrium for $t < 0$ with the switch clos The switch then opens at $t = 0$. Find an expression for the the voltage v(t), good for all time.



Systems Chapter 14 Study Guide

Transformers

A. Concepts Addressed By This Chapter

1. Mechanical, Electric, Hydraulic Transformers
2. Transformer ratio
3. Ideal Transformer
4' Equivalent Circuit Diagrams
5. Reflected Impedance

B. Introduction

For each system we have studied, we have defimed "pure" elements in which was lumped just one particular system parameter such as mass, inductance, elastance, etc. We also called these elements ideal if their behavior was linear in all cases. Basically, each element was defined to account for a form of energy in the system, either stored or dissipated. We now come to an ideal element found in almost all system types which neither stores or dissipates energy. Its importance lies in its ability to tranfer energy without loss while changing certain system variables. This element is called a "transformer". While the name transformer is most often used to denote magnetically coupled coils of the electric transformer, most system types have elements which perform the same operation: adjustment of values of both the across and through variables while maintaining their product (power) constant. Examples of common transformers include levers, gears, pulleys, magnetically coupled coils, hydraulically coupled pistons, and many others.

C. Instructional Objectives

A student mastering this material will be able to

1. Model physical systems which include transformers.
2. Given a transformer which is part of a system, determine the transformer ratio..
3. Include the proper transformer equivalent diagram in the system model.
4. Formulate system equations by using the concept of reflected impedance.
5. Determine the proper transformer ratio to achieve maximum power transfer.

D. Study Procedure

Read Chapter 14. Additional material can be found in references 9, 11.

Chapter 14
Transformers

1. Definition

An ideal transformer is a multiple-port system component which contains no energy storage elements, and no energy dissipative elements (Figure 1 shows a two-port transformer). It transfers energy among its ports with no energy loss. Since, in all systems, the law of conservation of energy must be satisfied, = power flow into the system of Figure 1 must be zero. In other words, power in = power out at any instant.

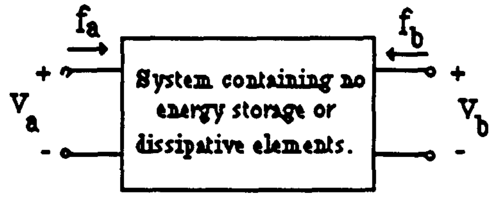


Figure 1
Net power flow into a system containing no energy storage or dissipative elements must be zero.

In the mechanical, electrical and fluid systems we have studied, power (energy flow) is the product of the across and through variables. Therefore, for the system of Figure 1, vafa represents power flowing into the system at port a, and vbfb represents power flowing into the system at port b. Since no energy can be stored or dissipated within the system, the sum of these power flows, (vafa + vbfb), must be zero at any instant. This can be written in the form of Equation 1:

$$vafa = \_ \; vbfb \qquad (1)$$

While this equation must hold true, it is not necessary that va and vb or fa and fb be equal.

Components which follow the relationship of Equation 1 exist in most systems. As was the case with the other system elements we have encountered, we can isolate this action and create a generalized ideal system element to represent it A major difference between the ideal transformer element and the other (R, L, C) ideal elements is that the transformer is a two-port device. That is, it has an input pair of terminals (a), and an output pair of terminals (b). There are, therefore, two across variables and two through variables associated with a transformer.

2. Mechanical Transformer: Lever

Translational Example: Lever (assumes all angles are

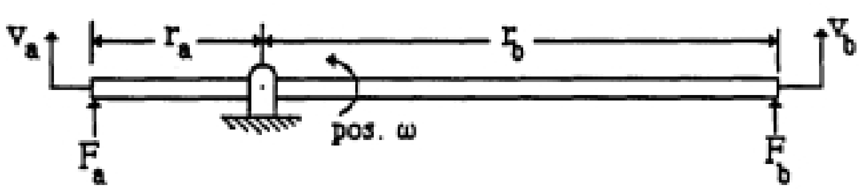


Figure 2
Lever is a rigid, massless bar which moves without friction.

If the lever moves,

$$\omega = \frac{-v_a}{r_a} = \frac{+v_b}{r_b} \qquad \text{so} \qquad \frac{v_a}{v_b} = -\frac{r_a}{r_b}$$

In addition, since the lever has no mass, moments about the pivot point must be zero:

$$-F_a\, r_a + F_b\, r_b = 0 \qquad \text{so} \qquad \frac{F_a}{F_b} = \frac{r_b}{r_a}$$

We define the ratio of the across variables as the "transformer ratio", η. Then,

$$\frac{v_a}{v_b} = \eta\,, \qquad \frac{F_a}{F_b} = -\frac{1}{\eta} \tag{2}$$

It can be seen that Equations 2 are consistent with the general definition of Equation 1.

An equivalent diagram composed of dependent sources can be used to represent the transformer described by Equations 2. Either diagram of Figure 3 satisfies these equations.

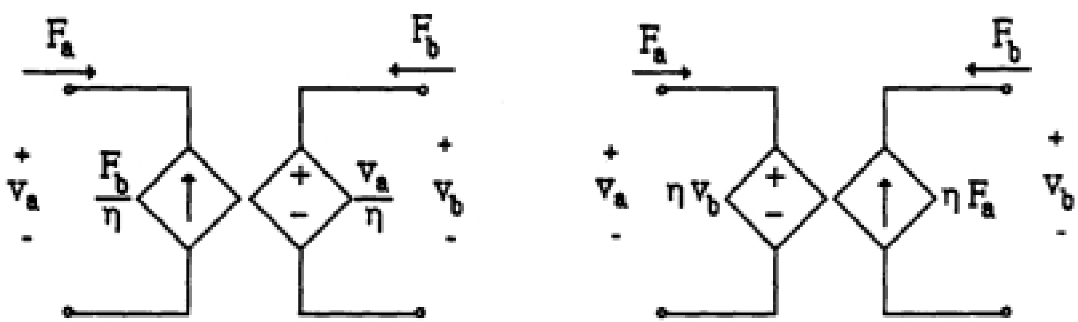


Figure 3
Equivalent diagrams for an ideal transformer.

A simpler diagram is also sometimes used to represent the ideal transformers of Figure 3. It is given in Figure 4. Here, only the essential information is supplied. This figure is understood to mean that the across variable on the a side is η times the across variable of the b side. The dots indicate which terminal orientation the given ratio (η) applies to. This diagram indicates that v_a as shown is η times v_b. If the dot on the right appeared at the lower terminal instead, it would mean that v_a is η times $-v_b$.

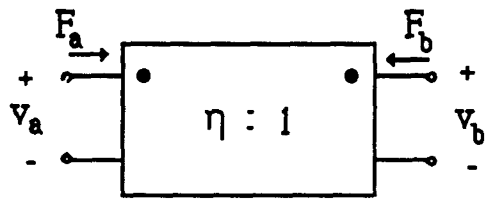
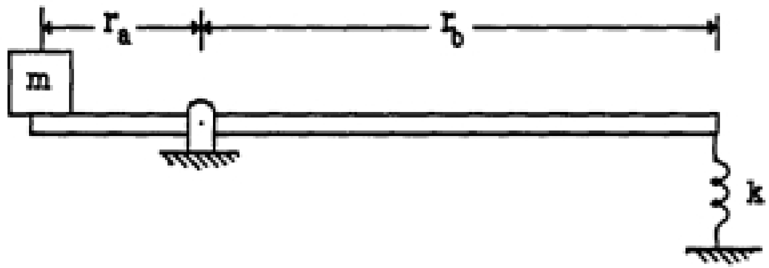


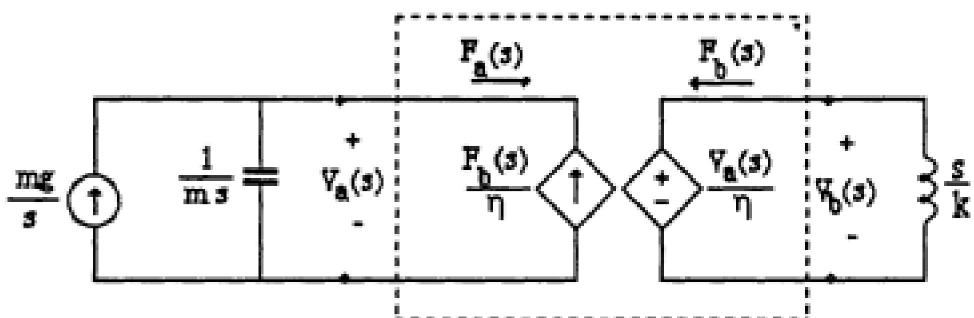
Figure 4
Symbol for an ideal transformer. η:1 is the ratio of across variables.

Example



Mass m is placed (not dropped) on the lever shown at $t = 0$, at a position r_a from the pivot. The other end of the lever is constrained by a spring of spring constant k. Determine the velocity of the mass for $t > 0$.

Solution: Since forces on each end of the this lever are downward, we will take this as the positive direction. The transformer ratio is $v_a/v_b = -(r_a/r_b) = \eta$. For this case, the equivalent diagram of the left panel of Figure 3 is best (although the other diagram can also be used to solve the problem. Draw a diagram of the connected elements in this system. Use the transformer diagram in place of the lever. The mass has the same velocity as the left end of the lever, so it should be connected from upper terminal a to reference. The force (weight of the mass) is applied also to this terminal:



A node equation written for the left side of the diagram yields:

$$V_a(s) = \left[\frac{mg}{s} + \frac{1}{\eta}F_b(s)\right]\frac{1}{s\,m}$$

An expression for Fb(s) in terms of Va(s) can be obtained from a loop equation on the right side of the diagram:

$$F_b(s) = -\frac{V_a(s)}{\eta \frac{s}{k}}$$

Substituting and rearranging gives the solution for velocity a as a function of

$$V_a(s) = \frac{g}{s^2 + \frac{k}{\eta^2 m}}$$

The corresponding time function (inverse transform) is

$$v_a(t) = \frac{g}{\omega} \sin \omega t, \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{\eta^2 m}}$$

3. Reflected Impedance:

In cases such as the previous example, where one side contains only passive elements (no independent sources), and the solution is for a variable on the other side, it is often convenient to "reflect" the impedance of these elements to the other side of the transformer. Consider a general passive impedance. ZL, connected on one side of a transformer, as shown in Figure 5:

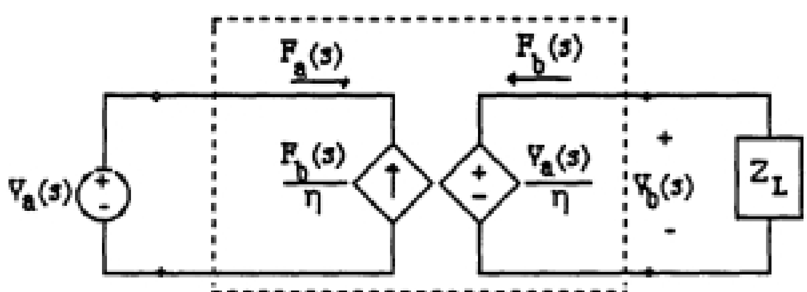


Figure 5
Passive secondary load, ZL, driven by a primary source, Va(s).

The effective impedance, Zin(s), that the source on the left works into is Zin = Va/Fa. Now, from the diagram of Figure 5,

$$F_a = -\frac{F_b}{\eta}, \qquad \text{and} \qquad F_b = -\frac{\frac{V_a}{\eta}}{Z_L}$$

Therefore,

$$Z_{in}(s) = \frac{V_a}{F_a} = \frac{V_a}{-\frac{F_b}{\eta}} = \frac{V_a}{\frac{\frac{V_a}{\eta}}{\eta Z_L}} = \eta^2 Z_L$$

So a passive load of impedance Z_L on one side of the transformer looks like an impedance of $\eta^2 Z_L$ (or $[1/\eta^2]\, Z_L$ if the sides are reversed) from the other side.

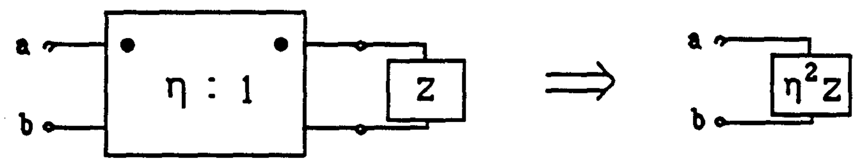
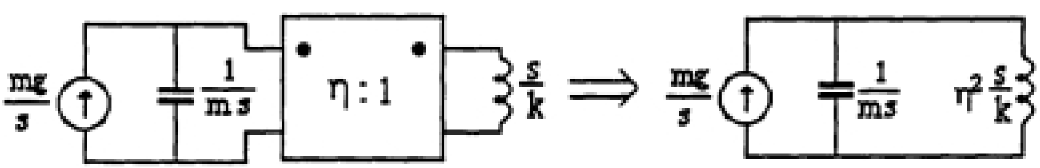


Figure 6
An impedance reflected across a transformer

The previous example could have been simplified by reflecting the spring load to the input *side* and thus eliminating the transformer from the calculations:



4. **Mechanical Transformer:** Gears

Suppose two disks with non-slipping edges (or two gears) are connected to external rotational system components which transmit torque and angular velocity as indicated in Figure 7.

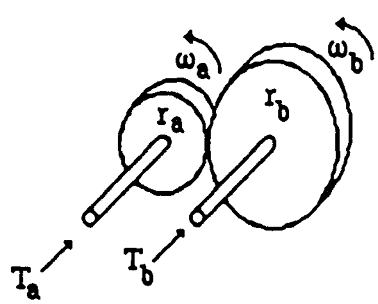


Figure 7
A rotational transformer

Assuming no slippage at the point of contact, both disks must have the same tangential velocity. That is, $\omega_a r_a = -\omega_b r_b$ where r_a and r_b are the respective disk radii. Note that the same will hold true, except possibly for the sign, for two disks connected by a belt. The across variable defines the transformer ratio so

$$\eta = \frac{\omega_a}{\omega_b} = -\frac{r_b}{r_a}$$

If the rotating elements are gears (which may also be connected by a chain, as in a bicycle), the transformer ratio is also equal to the ratio between the number of teeth of each gear.

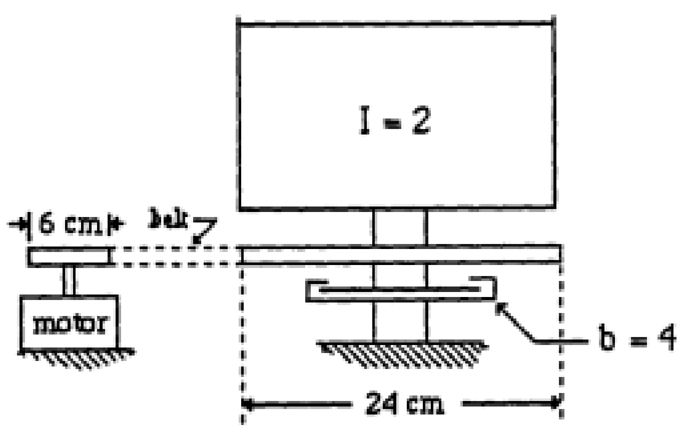
At the point of contact, assuming torque Ta applied to the shaft of a is positive, the force exerted by disk a on disk b is Fba = Ta/ra (downward). By Newton's 2nd law, an equal but opposite force must be returned by b to a at the same point: Fab = Tb/rb (upward). Since the magnitudes of these forces must be equal, Fab = Fba,

$$\frac{T_a}{r_a} = \frac{T_b}{r_b} \quad \text{which means that} \quad \frac{T_a}{T_b} = -\frac{1}{\eta}$$

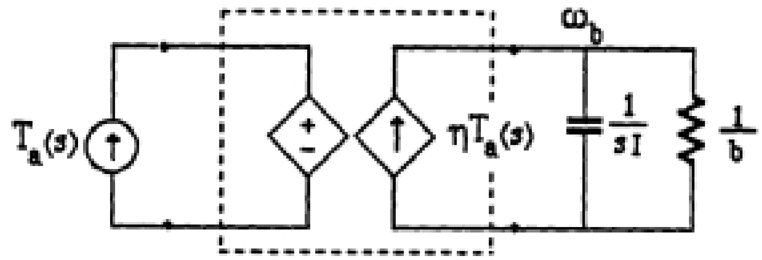
These equations relating the across and through variables of the mechanical rotational transformer are the same as those derived in Equation 1 for the mechanical translational transformer. Therefore, the equivalent diagrams of Figures 3 and 4 also apply to the rotational transformer.

Example:

The washing machine motor shown below is switched on at t = 0. From that time on, it delivers a constant amplitude, sinusoidal output torque: T =10 sin(2t) u(t). The motor, through belt driven pulleys, drives a rotational load consisting of a rotor of moment of inertia I = 2 Nms2, and viscous bearing friction, b = 4 Nms. Diameters of the pulleys are da = 6 cm, and db = 24 cm. Find the angular velocity of the rotor as a function of time.



Solution: Use the equivalent diagram from the right-hand panel of Figure 3. To the input (left side), connect the driving motor which is an ideal through source, Ta(s). On the right, connect the rotational inertia, impedance 1/sI and the damping element, R=1/b.



The velocity, ω_b, at the top node on the right can be found from a vertex equation:

$$\eta\, T_a(s) = \omega_b(s)\left[b + sI\right]$$

The sine-wave input torque has the Laplace transform $T_a(s) = 32[\ 2/(s^2+4)\ ]$.

Solving the above equation for $\omega_b(s)$ yields:

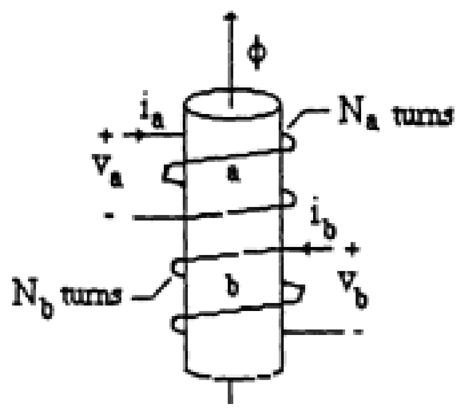
$$\omega_b(s) = \frac{8}{(s^2+4)(s+2)} = \frac{1}{s+2} - \frac{s-2}{s^2+4}$$

The inverse transform is

$$\omega_b(t) = [\ e^{-2t} - \cos 2t + \sin 2t\ ]\ u(t).$$

5. Electric Transformer

Example: Magnetically coupled coils. Consider a cylindrical core on which are wound two resistanceless coils, a and b. Coil a has N_a turns and coil b has N_b turns. When current flows in either coil, magnetic flux, ϕ, is produced in the core. If all flux lines link all turns, we say the transformer is "unity coupled".



The voltage induced in a coil of N turns, linked by a flux of ϕ webers, is given by Faraday's Law:

$$v = \frac{d\lambda}{dt} = N\frac{d\phi}{dt}$$

Since the same flux, ϕ, links all turns of both coils, the respective voltages induced in the coils are:

$$v_a = N_a \frac{d\phi}{dt} \qquad \text{and} \qquad v_b = N_b \frac{d\phi}{dt}$$

We have defined the transformer ratio, η, as the ratio of the across variables. Therefore, for this electric transformer,

$$\frac{v_a}{v_b} = \frac{N_a}{N_b} = \eta$$

The total flux in the core is the sum of that produced by each current:

$$\phi = k\,(i_a N_a + i_b N_b)$$

where k is a proportionality constant. If the transformer is ideal, then, by definition, no stored energy is allowed. (Magnetic flux indicates the presence of a magnetic field which is a form of stored energy). Therefore, ϕ must be always zero. Since $0 = i_a N_a + i_b N_b$,

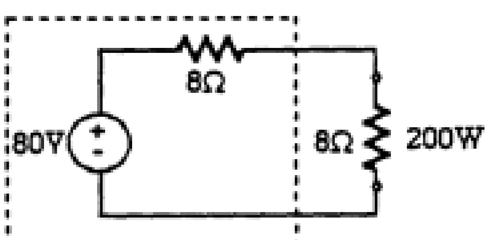
$$\frac{i_a}{i_b} = -\frac{1}{\eta}$$

This means that if current is driven through one winding, tending to produce a magnetic flux in the core, a current must flow in the other winding of such magnitude and direction as to produce an equal but opposite flux, resulting in a net flux always of zero. Once again, any of the equivalent diagrams of Figures 3 or 4 may be used for the ideal electric transformer.

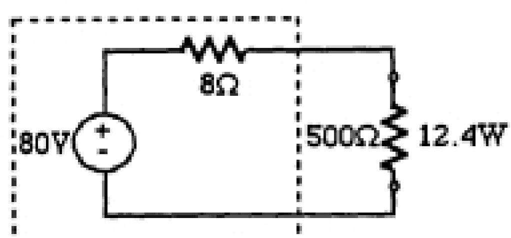
Example:

A public address amplifier rated at 200 watts output, has an internal impedance of 8Ω (usually referred to as the amplifier's "output impedance"). By the maximum power transfer theorem, this amplifier will deliver maximum output power to a "matched" 8Ω load, for example, an 8Ω loudspeaker. The amplifier's output circuit may be represented by a Thevenin equivalent, as shown below. We know the amplifier can deliver 200W, so its Thevenin voltage can be found to be 80V.

We want to use this amplifier to drive a 500 ohm input impedance strip-chart recorder. Using the same Thevenin equivalent for the amplifier now connected to this 500Ω load, we find that the power delivered is only 12.4 W. The decrease in power output is caused by the fact that the load is no longer matched to the amplifier. How can we achieve maximum power transfer between this amplifier and the 500Ω load?



Matched

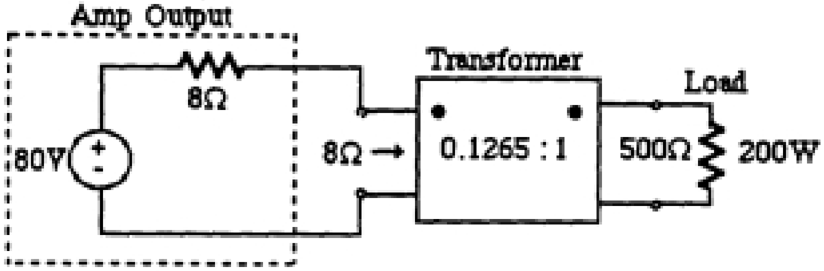


Not Matched

Solution:

The transformer must make 500Ω look like 8Ω to the amplifier. Looking at Equation 2 and

Figure 6, we see that a transformer can make an impedance, Z, look like $\eta^2 Z$. We need a transformer with η such that $\eta^2(500) = 8$, or $\eta = 0.1265$. This can be approximated by a ratio such as 10:79. A 10:79 transformer would make 500Ω look like 8.01Ω. The power delivered to the 500Ω load in this case would be approximately 200W.



6. **Hydraulic Transformer**

In hydraulic systems, the across and through variables are pressure, P, and fluid volumetric flow rate, Q. The fluid system of Figure 8 illustrates one way of transforming between these variables. Pistons a and b have faces with areas Aa and Ab, respectively, and are connected by a rigid rod. The space between the pistons is open to atmosphere, or to some fluid reservoir of constant pressure, P0. As was the case with our previous transformer examples, we assume rigid, massless, frictionless components.

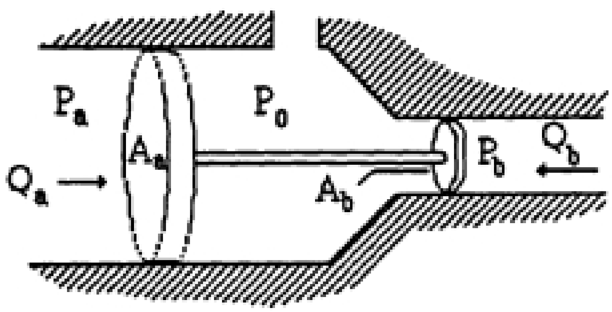


Figure 8
Hydraulic Transformer

The force exerted by the fluid on side a against piston face a is $(P_a - P_0)A_a$, directed to the right. Similarly, the force exerted by the fluid on side b against piston face b is $(P_b - P_0)A_b$, directed to the left. Since the piston system is massless, these forces must be equal. Therefore, $(P_a - P_0)A_a = (P_b - P_0)A_b$. After cancellaton of the P_0 terms, the transformer ratio, defined as the ratio of the across variables, may be found:

$$\frac{P_a}{P_b} = \frac{A_b}{A_a} = \eta$$

If the pistons move, they move at the same velocity. Assume motion to the right with velocity v. This velocity is related to the flow rates by

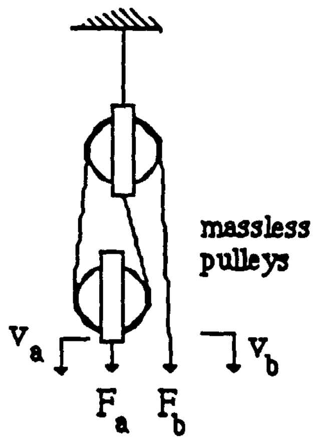
$$v = \frac{Q_a}{A_a} = -\frac{Q_b}{A_b}$$

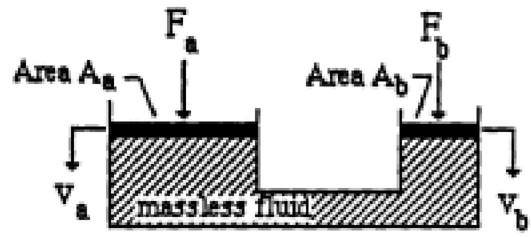
The ratio of flow rates on sides a and b is then

$$\frac{Q_a}{Q_b} = -\frac{A_a}{A_b} = -\frac{1}{\eta}$$

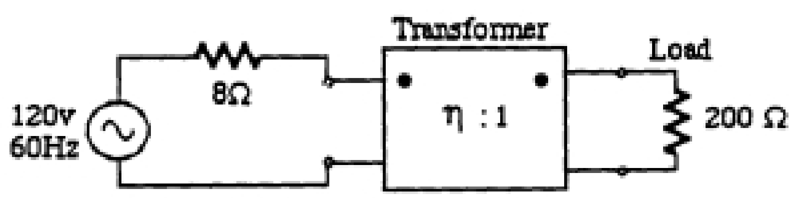
So this hydraulic transformer has the same basic relationships found for the other transformers we have encountered, and the equivalent diagrams of Figures 2 and 3 may be used for this example as well.

1. For each diagram below, determine the transformer ratio, h, and sketch an equivalent diagram using dependent souces' Ans: η= -2, -(AblAa).



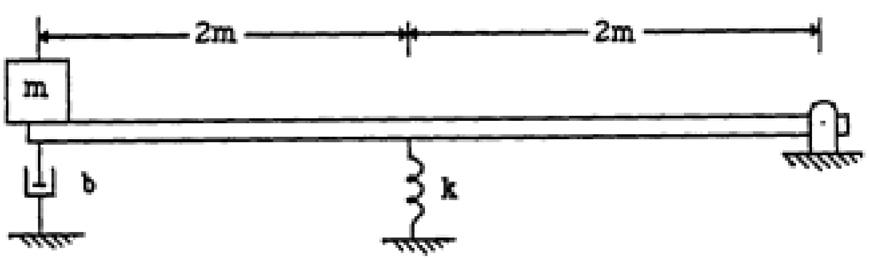


2. What turns ratio is needed to get maximum power to the load resistor? What is the value of this maximum power? Ans: η = 0.2, 450W

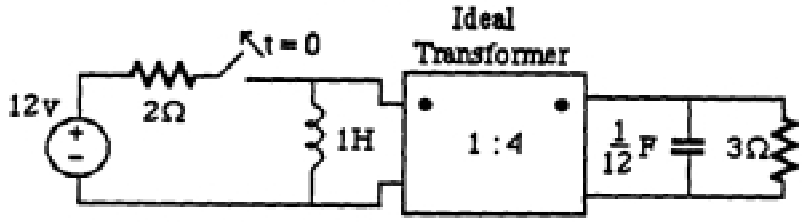


3. Find the velocity of a mass placed (gently) on the left side of the (massless) lever. Assume angles are small. g = 9.8. Ans: 1.633(e^{-2t} - e $^{-8t}$) u(t)

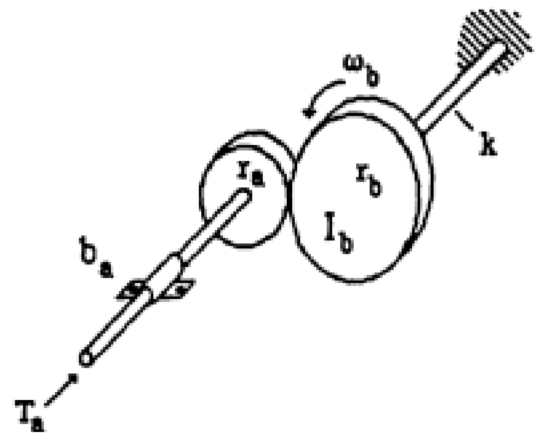
m = 1/80 kg, b = 1/8 Ns/m, k = 4/5 N/m



4. The system is in equilibrium when, at t = 0, the switch is opened. Find the voltage across the 3Ω resistor for t > 0. (Ans: $18\,(e^{-3t} - e^{-t})\,u(t)$)



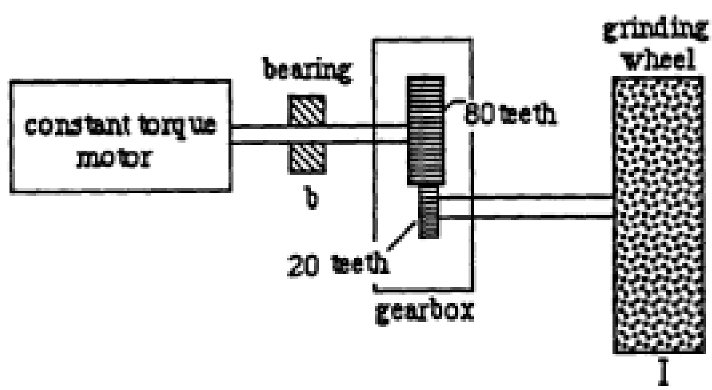
5. Determine the transfer function, H(s), for the system below. The input is torque, Ta, applied to shaft a, and the output is the angular velocity of rotor b. Shaft a turns against viscous rotational bearing friction, ba. Wheel a may be considered massless, but wheel b has moment of inertia Ib. Wheel b is attached to the reference frame through a flexible shaft of spring constant, k.



6. A constant torque motor drives a grinding wheel through a gearbox as shown. The motor produces a constant output torque of 40 Nm. The shaft is subject to viscous bearing friction, b = 0.5 Nms. The motor drives a gear with 80 teeth which drives a second gear with 20 teeth. The moment of inertia, I, of the grinding wheel is 0.75 kg.

(a) Sketch a model of this system using ideal elements.

(b) If the system is at rest when the motor is switched on at t = 0, find the angular velocity of the wheel, ω(t), for t > 0.



Systems Chapter 15 Study Guide

Sinusoidal Steady State

A. Concepts Addressed By This Topic

1. Phasor representation of sinusoids.
2. Impedance, $Z(j\omega)$, Admittance, $Y(j\omega)$, Transfer Function, $H(j\omega)$.
3. Effective value of a sinusoid.
4. Power calculations in steady-state AC electrical systems, including power factor, and complex power

B. Introduction

When a linear system is driven by one or more sinusoidal forcing functions of constant amplitude and frequency, the system is said to be operating in the "sinusoidal steady-state". When an electric system operates in this mode, it is often referred to as an "AC" system (AC stands for alternating current). The behavior of systems in this state is so important in many fields of engineering that special techniques have been devised to handle calculations with these systems.

We have already seen that sinusoidal variables are best handled mathematically by the use of complex phasor notation. If a sinusoidal across or through variable is applied to any of the three . ideal element types we have studied, the forced response is also a sinusoid of the same frequency. Therefore, using complex phasors, the ratio of across to through variables (voltage to current) for any element or group of elements is a complex constant defined as the "Impedance", $Z(j\omega)$. Because impedances combine like resistances in series or parallel, we can use impedance combinations to simplify systems. The use of the impedance concept and complex phasor mathematics permits the algebraic analysis of sinusoidal steady state systems.

Probably the most important application of the analysis methods of this topic is the field of electric power distribution. In the United states, most electric power systems operate at 60 Hz. Voltage and current in such systems are measured not according to their maximum values, but in terms of a quantity related to power called "effective" or "rms" value. This is the value found on the nameplate of electrical equipment. Since these across and through variables are sinusoids and not necessarily in phase, power in AC systems is also time-varying. Average power is of more importance in most AC systems than is instantaneous power. The average power dissipated per cycle (P) is a measure of the power delivered to resistance in the system. Reactive power (Q) is the average of power flow to and from L and C elements in the system (it is not dissipated). The two are sometimes combined in a complex number, $S = P + jQ$, called complex power.

C. **Instructional Objectives**

A student who masters this material will be able to

1. Use the concepts of Impedance and Admittance to solve AC steady state systems.
2. Find the effective or rms value periodic waveforms.
3. Calculate average, reactive, and complex power in a system'
4. Use complex power to recommend power factor correction elements for

E4 Systems
Chapter 15
Sinusoidal Steady State

We have seen that signals of the form est occur frequently in the solution of system equations. Another time function which is very common to linear systems is the sinusoid. In systems of second or higher order, sinusoids are often part of the natural response. The sinusoid is also a commonly used forcing function (e.g. all ac electric devices). In addition, by a method called Fourier Analysis, it is possible to express any periodic signal as the sum of sinusoids of different frequencies.

Because sinusoidal signals are so common, a special method has been devised to handle these functions. Each sinusoid is represented by a complex number called a "phasor". The algebraic combination of sinusoids then becomes an exercise in the algebra of complex numbers. After a result is found in phasor form, it must be converted back to a sinusoidal function of time.

1. Review of Complex Algebra

A complex number is simply a number pair. Unfortunately the names "real part" and "imaginary part" have been adopted as the names of the two parts of the complex number. The names do not mean that either part is more authentic than the other.

Since the complex number has two parts, the most convenient way to represent it graphically is as a point in a plane. We can use the horizontal (real) axis to indicate the first number of the pair (the real part), and the vertical (imaginary) axis to indicate the second number of the pair (the imaginary part). The figure shows how the complex number (5, 3) can be depicted on the complex plane. (Here, italics are used to designate the imaginary part.)

imaginarv

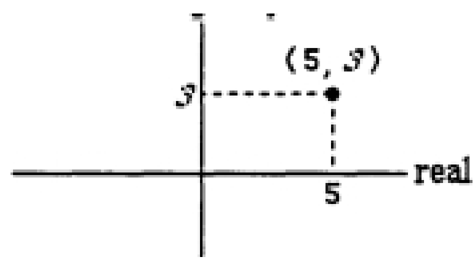


Figure 1 Graphical Representation of a Complex Number

In order to define mathematical operations which involve both parts of complex numbers, we need to express both parts in terms of one number type. To do this, we define an operator, j, which has the ability to convert a real number to an imaginary number of the same magnitude. j operating on real number, a, converts it to imaginary number, a. That is, j a = a

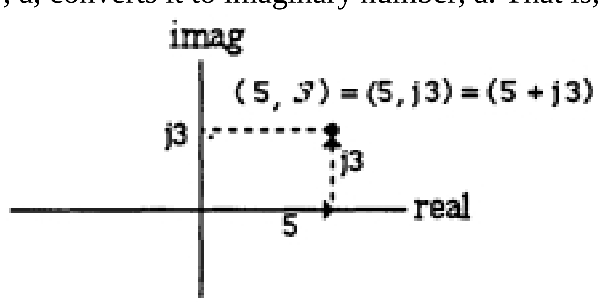


Figure 2 Use of Operator j

So complex number (5, 3) can be expressed (5, j3). Graphically (Figure 2) we see that this is also the same as 5 + j3. That is, 5 units in the real direction plus 3 units in the imaginary direction.

The effect of operator j is a rotation of the number operated upon by 90° in the complex plane. Figure 3 shows this effect of operator j , rotating the point at 4 on the real axis to the point 4 , or j4 on the imaginary axis.

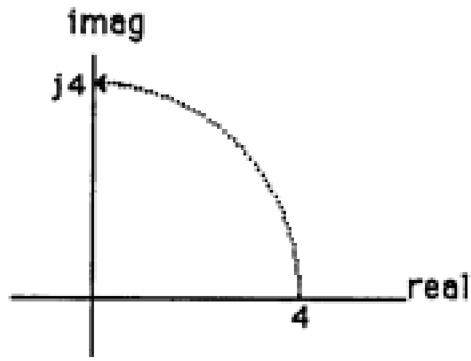


Figure 3 j Operator Rotates Point 90^0

Suppose we operate a second time with j:

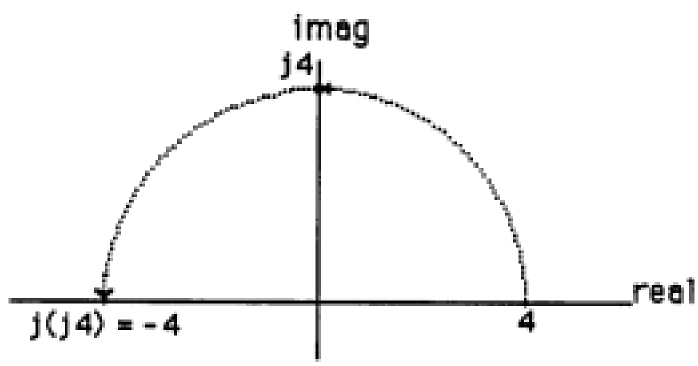


Figure 4 j^2 Equivalent to -1

Figure 4 indicates that j(j4) rotates the point twice 90^0 to the negative real axis where it falls on the value -4. So operating with j twice (j^2) is equivalent to multiplication by -1. Even powers of j can therefore be converted to +1 or -1. Odd powers of j can be converted to +j or -j. Some exaples: $j^4 = 1$, $j^3 = -j$, $j^{-1} = -j$.

Since use of the j operator permits both the real and imaginary parts of the complex number to be expressed with real numbers, we may now apply all the usual rules of algebra to combinations of these parts. (We will use bold face type to indicate a complex number.) Given that complex number $\mathbf{Z_1} = a + jb$, and $\mathbf{Z_2} = c + jd$, then

$$\mathbf{Z_1} + \mathbf{Z_2} = a + c + j\,(b + d)$$
$$\mathbf{Z_1} - \mathbf{Z_2} = a - c + j\,(b - d)$$
$$\mathbf{Z_1}\,\mathbf{Z_2} = ac + jbc + jad + j^2bd = (ac - bd) + j\,(bc + ad)$$
$$\frac{\mathbf{Z_1}}{\mathbf{Z_2}} = \frac{a + jb}{c + jd}$$

To separate real and imaginary parts of $\mathbf{Z_1}/\mathbf{Z_2}$, multiply by (c - jd)/(c - jd). Note that (c + jd)(c - jd) = $c^2 + d^2$. This process is called rationalizing the denominator.

$$\frac{a + jb}{c + jd}\,\frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

Polar Form

If we use a two-dimensional graph to plot complex numbers, it is possible to locate a given point given its real and imaginary parts. A second way of locating the same point is by specifying a distance from the origin (the magnitude M of the complex number), and a direction (the angle θ measured from the positive real axis). A shorthand way of designating magnitude M and angle angle θ is M∠θ.

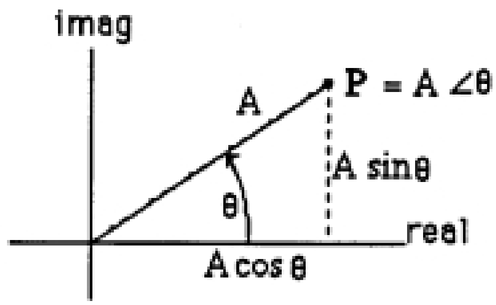


Figure 5 Complex Number in Polar Form

By geometry, we can derive the following relations using Figure 5 (absolute value bars indicate magnitude of the complex number):

$$A = \sqrt{a^2 + b^2} = |\mathbf{Z}|$$

$$\theta = \tan^{-1}\frac{b}{a} \tag{1}$$

Inversely,

$$a = A\cos\theta$$
$$b = A\sin\theta \tag{2}$$

Euler's Identity

A mathematical relation called Euler's Identity is used extensively in systems and signal analysis. It enables the conversion of harmonic functions between exponential and forms. Consider the complex number $\mathbf{P} = A\angle\theta$ of Figure 5:

$$\mathbf{P} = A \cos\theta + j A \sin\theta \qquad (3)$$

$$\frac{d\mathbf{P}}{d\theta} = A\left[-\sin\theta + j\cos\theta\right] = j\,A\left[\cos\theta + j\sin\theta\right] = j\mathbf{P}$$

$$\frac{d\mathbf{P}}{\mathbf{P}} = j\,d\theta \qquad (4)$$

$$\int \frac{d\mathbf{P}}{\mathbf{P}} = \int j\,d\theta$$

$$\ln \mathbf{P} = j\theta + K \qquad (5)$$

We can evaluate K by choosing any convenient angle for θ (such as 0^o or 90^o). If $\theta = 0$, $\mathbf{P} = A\cos 0 + jA\sin 0 = A$. Then Equation 5 says that

$$K = \ln A \qquad (6)$$

So, $\ln \mathbf{P} = \ln A + j\theta$

Taking antilogs,

$$\mathbf{P} = e^{(\ln A + j\theta)} = e^{\ln A}\, e^{j\theta} = A\,e^{j\theta} \qquad (7)$$

Substituting from Equation 3,

$$\mathbf{P} = A\cos\theta + jA\sin\theta = A\,e^{j\theta} \quad (\text{ or } A\angle\theta) \qquad (8)$$

Equation 8 is known as Euler's Identity.

By changing signs on θ and repeating the above development, we can show that

$$A\cos\theta - j\,A\sin\theta = A\,e^{-j\theta} \qquad (9)$$

Addition and subtraction of Equations 8 and 9 yield other

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad (10)$$

2. **Phasor representation of sinusoids**

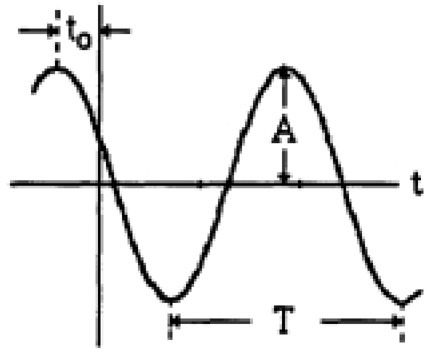


Figure 6 General Sinusoid

A general expression for a sinusoid in any phase position is

$$x(t) = A \cos \omega(t + t_0) = A \cos (\omega t + \theta_0) \tag{11}$$

where A is amplitude or peak value
T is the period (seconds per cycle, s^{-1})
ω is radian frequency (radians per second, s^{-1})
t_0 is time shift (seconds)
$\theta_0 = \omega t_0$ is phase shift (degrees or radians).

Every complete cycle contains 2π radians. Therefore, the number of radians per second is equal to 2π times the number of cycles per second:

$$\omega = 2 \pi f \tag{12}$$

where f, the number of cycles per second is the reciprocal of the period, T, the number of seconds per cycle' f is called the cyclic frequency.

By Euler's Identity (Equations 8 and 10), we

$$A \cos (\omega t + \theta) = \frac{1}{2} A e^{j(\omega t+\theta)} + \frac{1}{2} A e^{-j(\omega t+\theta)} \tag{13}$$

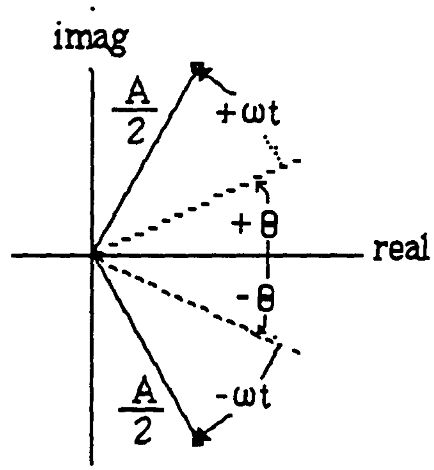


Figure 7 The terms of Equation 13

Figure 7 shows the two complex terms of equation 13 at a particular instant in time. The solid lines (phasors) representing the terms, rotate with angular velocities + ω and - ω. If the two are added at any point in time, the result is a real number, A cos(ωt + θ), because the imaginary parts always cancel The dotted lines represent the postion of the phasors at the instant t = 0.

Replacement of sinusoidal functions such as A cos (ωt + θ) by complex exponential functions such as $(A\,e^{j\theta})(e^{j\omega t}) = A \angle (\omega t + \theta)$ may not appear at this moment to be a simplification, but you will find in the following examples that it will simplify calculations involving multiple sinusoids of the same frequency.

Suppose we want to add a(t) + b(t), where $a(t) = A \cos(\omega t + \theta_A)$, $b(t) = B \cos(\omega t + \theta_B)$. (Note that a and b are sinusoids of the same frequency, a necessary requirement for the method we are deriving here.) Euler's Identity allows the conversion of a(t) and b(t) to exponential form as was done in Equation 13. So for each sinusoid, we get two rotating complex values of magnitude A/2 and B/2, respectctvely.

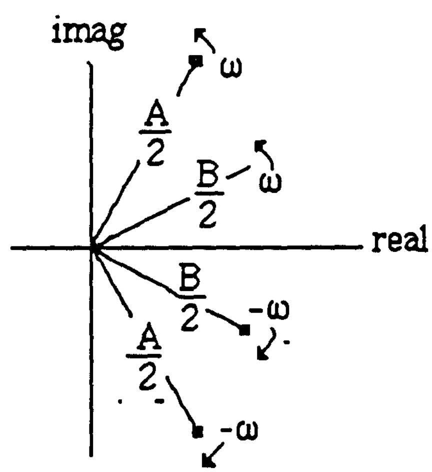


Figure 8 Phasor Components of a(t) and b(t) of Example 1

Figure 8 shows these two sinusoids represented as phasor pairs, in the position they hold at $t = 0$. This is the time usually chosen for a "snapshot" of the rotating lines, but remember that all are actually rotating at angular velocities ω or $-\omega$. We need to add these four complex numbers to get the result of $a(t) + b(t)$. The addition can be simplified if we make use of the following facts:

1. At all times, the imaginary parts of each phasor pair cancel.
2. The real parts of each phasor pair are equal.

Therefore, if we add just the two positively rotating phasors after doubling their magnitude, and throw away the imaginary part of the result of this addition, we will get the same result as would be obtained by adding all four.

General procedure for algebraic combinations of sinusoid of the same frequencey:

1. Convert all sinusoids to cosine form.
2. Write each sinusoid as a single phasor with magnitude equal to the peak value of the sinusoid and angle equal to its phase angle at $t = 0$.
3. Perform the necessary algebra using the complex numbers of the phasor representation.
4. Convert the resulting phasor back to an ordiary cosine time function.

Example 1:

Find $a(t) + b(t)$ if $a = 10 \cos(\omega t + 20^{\circ})$ and $b = 5 \cos(\omega t + 60^{\circ})$.

Solution: Represent these functions by two positively rotating phasors:

$\mathbf{A} = 10 \cos 20^{\circ} + j\,10 \sin 20^{\circ} = 9.4 + j\,3.42$

$\mathbf{B} = 5 \cos 60^{\circ} + j\,5 \sin 60^{\circ}. = 2.5 + j\,4.33$

$\mathbf{A} + \mathbf{B} = 11.9 + j\,7.75 = 14.2\angle 33^{\circ}$

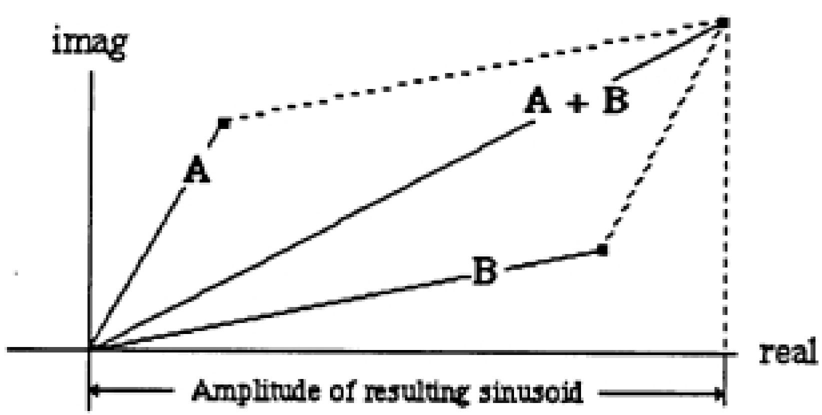


Figure 9 Phasors of Example 1

So the result of the addition is $a(t) + b(t) = 14.2 \cos(\omega t + 33^{\circ})$

Example 2:

Given $a(t) = 10 \cos(\omega t + 53.1^o)$ and $b(t) = 5 \cos(\omega t - 30^o)$. Find a(t) - b(t)

Solution: Phasor representation of $10 \cos(\omega t + 53.1^o)$ is

$\mathbf{A} = 10\angle 53.1^o = 6 + j\,8.$

Phasor representation of $5 \cos(\omega t - 30^o)$ is

$\mathbf{B} = 5\angle 30^o = 4.33 - j\,2.5$

To get the difference, subtract reals and imaginaries

$10\angle 53.1^o - 5\angle 30^o = (6 - 4.33) + j\,(8 + 2.5) = 1.67 - j\,10.5 = 10.6\angle 81^o$

So the solution is $a(t) - b(t) = 10.6 \cos(\omega t + 81^o)$.

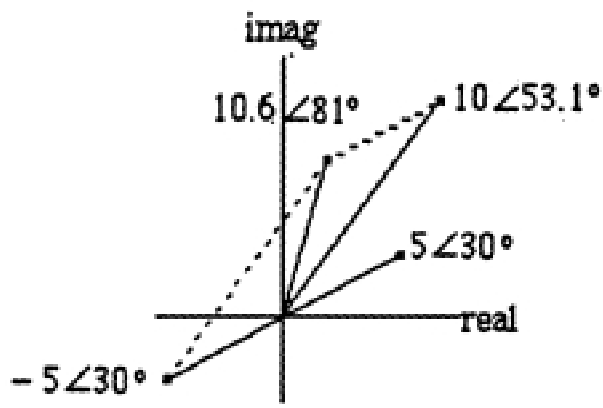


Figure 10 Phasor Diagram of Example 2

Example 3: $a(t) = 10 \cos \omega t$, $b(t) = 20 \sin(\omega t + 50^o)$ Find a(t) + b(t).

Here, it is important to first convert the sine function to a cosine. This is necessary because the phasor method we have adopted defines the angle such that its <u>cosine</u> yields the real part of the phasor and therefore the answer. The identity we need is:

$$\sin \omega t = \cos(\omega t - 90^o).$$

This means that $\sin(\omega t + 50^o)$ can be replaced by $\cos(\omega t - 40^o)$. The phasors are then $\mathbf{A} = 10\angle 0^o$, and $\mathbf{B} = 20\angle{-40^o}$. Adding these in the manner illustrated above yields $28.4\angle{-27^o}$, which is the phasor representation of the sum of a and b:

$$a(t) + b(t) = 28.4 \cos(\omega t - 27^o).$$

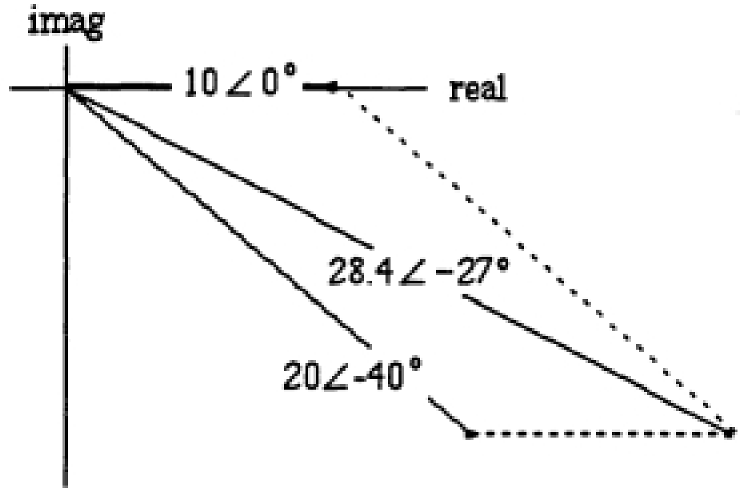


Figure 11 Phasor Diagram of Example 3

3. System Elements in Sinusoidal Steady-State

If a sinusoidal forcing function is applied to any of our basic system elements (A, T, D), a sinusoidal forced response results. In Figure 12, we apply v(t) = Vm cos wt to each element, and wait for steady state conditions. (Steady state means that all transient effects due to switching the signal on, etc.,have died out.). When a linear circuit driven by sinusoidal voltage and/or current sources reaches steady state, all voltages and currents in all elements are sinusoidal and of the same frequency.

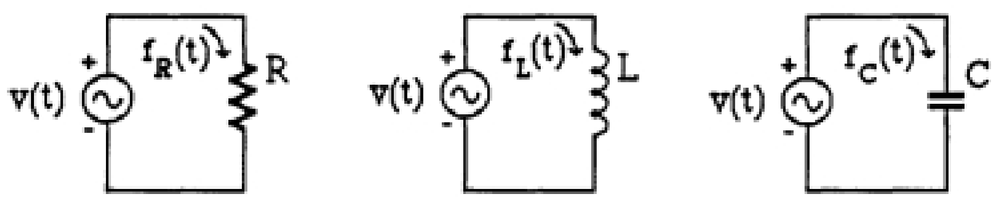


Figure 12 Sinusoidal forcing function, v, applied to the three basic system elements

Application of $v = V_m \cos \omega t$ to the defining equations of R, L, and C result in the following solutions for steady-state flow:

$$f_R = \frac{V_m}{R}\cos \omega t \qquad f_L = \frac{1}{L}\int V_m \cos \omega t \, dt \qquad f_C = C\frac{dv}{dt}$$

$$= \frac{V_m}{\omega L}\sin \omega t \qquad = -\omega C V_m \sin$$

$$= \frac{V_m}{\omega L}\cos(\omega t - 90°) \qquad = -\omega C V_m \cos(\omega t + 90°) \tag{14}$$

To simplify some of the notation and descriptions at this point, we will use the electrical system variable names, voltage and current, to refer to the across and through variables of our examples. This in no way restricts the methods presented here from application to any linear system.

Figure 13 depicts plots of current and voltage waveforms associated with each of the three electric circuit elements. Note that in the resistor, the voltage and current are in phase (pass through their max and min values at the same time), while the other two elements cause either a + or - 90° phase shift between voltage and current. In the inductor, the current has its maximum 90° the voltage has its maximum. We say that the inductor current lags the inductor voltage by 90°. For the capacitor, the current has its maximum 90° <u>before</u> the voltage. Therefore, we say that in the capacitor, current leads the capacitor voltage by 90°.

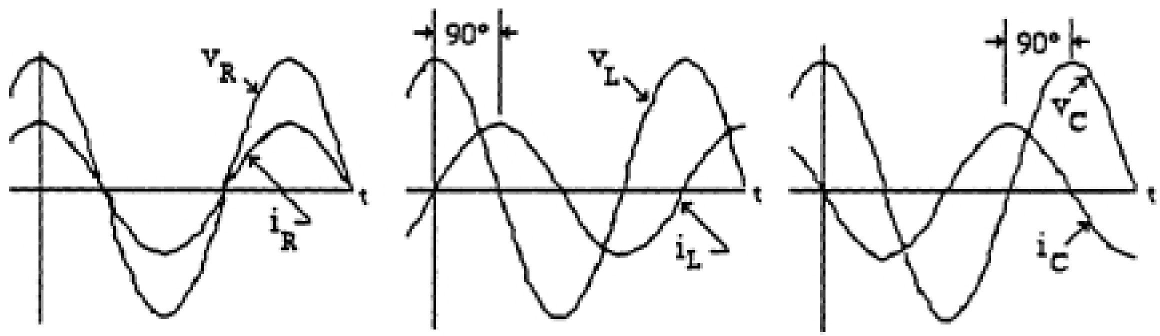


Figure 13 Sinusoidal current and voltage time reIationships in the basic electric elements. Left panel, resistor: no phase shift. Center panel, inductor. current lags voltage by 900' Right panel, capacitor. current leads voltage by 90°

The same results can be (more easily) obtained using phasor techniques. If sinusoidal voltage v(t) is represented by a rotating phasor, $\mathbf{V} = V_m e^{j\omega t}$, its derivative with respect to time is just $j\omega V_m e^{j\omega t}$ which is $j\omega\mathbf{V}$. also, if we integrate, $V_m e^{j\omega t}$ with respect to time, the result is $\frac{1}{j\omega} V_m e^{j\omega t} = \frac{\mathbf{V}}{j\omega}$. Therefore, the phasor expressions for voltage and current in pure R, L, C elements are:

$$\mathbf{V} = V_m\angle 0^\circ = V_m + j\,0$$
$$\mathbf{I}_R = \frac{V_m}{R}\angle 0^\circ = \frac{V_m}{R} + j\,0$$
$$\mathbf{I}_L = \frac{V_m}{\omega L}\angle -90^\circ = 0 - j\frac{V_m}{\omega L} = \frac{V_m}{j\omega L}$$
$$\mathbf{I}_C = \omega C V_m \angle 90^\circ = 0 + j\,\omega C V_m = \frac{V_m}{\frac{1}{j\omega C}} \qquad (15)$$

Figure 14 illustrates the relative phasor positions of V, IR, IL, and IC.

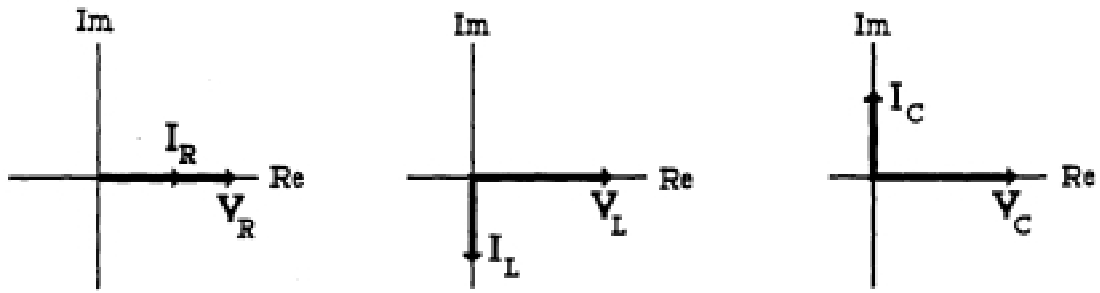


Figure 14 Phasor current and voltage relationships for the basic circuit elements

Impedance and Admittance

Impedance in AC electric circuits is defined as the ratio of phasor voltage to phasor current. The symbol Z is used to denote impedance. In general, the value of the impedance is a complex number, since it is the ratio of two complex phasors. The reciprocal of impedance has the name admittance. The symbol for complex admittance is Y.

When the term impedance is used in system analysis, it usually stands for the ratio of the effort variable to the flow variable, with each expressed as a function of s or of jw. The designation "effort" and "flow" is an alternate way of classifying the variables of a system. It was discussed in Section 10 of Chapter 5. For all but mechanical systems, the effort variable is the across variable and the flow variable is the through variable. For mechanical systems, the corresponding designations are reversed'

Since we have just determined that these ratios are constants for each element, each element's impedance is also a constant:

$$\text{Impedance,}\quad \mathbf{Z} = \frac{\text{phasor voltage}}{\text{phasor current}}$$

$$\boxed{\mathbf{Z}_R = R \qquad \mathbf{Z}_L = j\omega L \qquad \mathbf{Z}_C = \frac{1}{j\omega C}}$$

$$\text{Admittance,}\quad \mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\text{phasor current}}{\text{phasor voltage}}$$

$$\boxed{\mathbf{Y}_R = \frac{1}{R} \qquad \mathbf{Y}_L = \frac{1}{j\omega L} \qquad \mathbf{Y}_C = j\omega C} \tag{16}$$

IMPORTANT: Before applying loop or node methods or parallel/series combinations of elements, always start every sinusoidal steady state problem by labeling each element with its impedance (or admittance), using the forms given above.

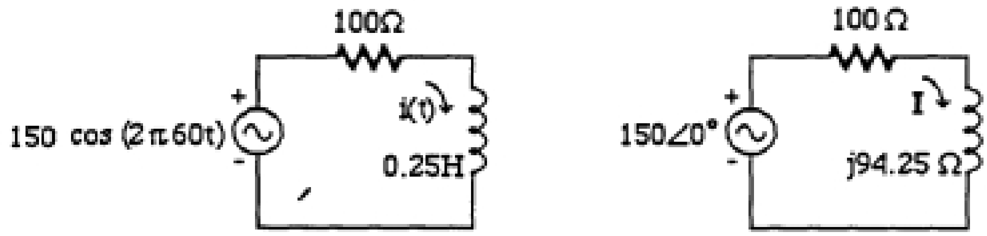
Impedances can be combined as were resistances in DC circuits: series impedances add, and parallel impedances can be combined using product-over-the-sum or reciprocal addition. The only difference is that the numbers are now complex Note that when elements of different types are connected together, the phase angle of the combined impedance may be any angle from -90° to + 90°. For parallel circuits, admittance units may be more convenient to use, as were conductance values in DC circuits.

Special names and symbols are given to the real and imaginary parts of the electric AC impedance and admittance quantities:

Re(Z): Resistance, R	Re(Y): Conductance, G
Im(Z): Reactance, X	Im(Y): Susceptance, B
$Z = R + jX$	$Y = G + jB$

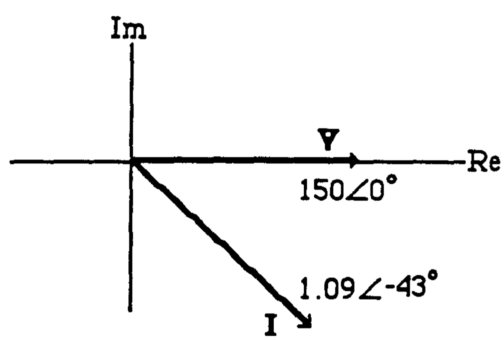
Note that in general, R and G are not reciprocals, nor are X and B.

<u>Example</u>: At a frequency of 60 Hz, determine the impedance of a series combination of a 100Ω resistor and a 0.25H inductor. What current would this combination draw from a 60 Hz voltage source with peak value 150 volts? (Circuit on the left, below.)



<u>Solution:</u> Redraw the circuit using impedances and phasor notation (right side, above). The frequency of 60 Hz is $\omega = 2\pi f = 2\pi 60 = 377$ r/s. Therefore $\mathbf{Z}_L = j\omega L = j(94.25)\Omega$. Since the elements are in series, $\mathbf{Z} = \mathbf{Z}_R + \mathbf{Z}_L = 100 + j\,94.25 = 137.4\angle 43.3°$

The phasor current is $$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{150\angle 0°}{137.4\angle 43.3°} = 1.09\angle{-43.3°}$$



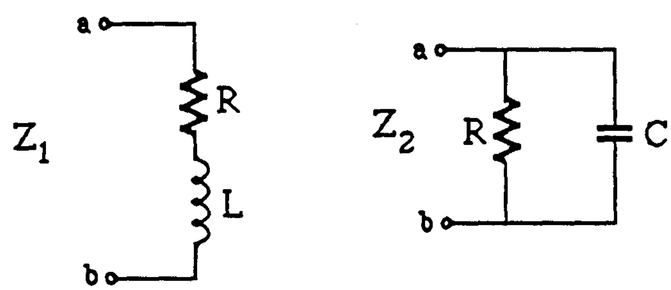
A quick sketch confirms what we expect to see in an inductive circuit: the current lags the voltage (by 43.3°). From the phasor current, we can write an expression for the actual current as a function of time in the RL circuit:

$$i_{RL} = 1.09 \cos(377t - 43'3')$$

If numerical values are not given, algebraic expressions for the impedance of element combinations can be determined using the same series/parallel combination techniques applied previously.

Example:

Determine the impedance, Z, of each circuit between terminals a-b:



Solution: For the series combination of Z1, simply add element impedances. The result may be converted to polar form

$$Z_1 = Z_R + Z_L = R + j\omega L = \sqrt{R^2 + \omega^2 L^2}\angle \tan^{-1}\left(\frac{\omega L}{R}\right)$$

For the parallel connection of Z2, use product over the sum:

$$Z_2 = \frac{Z_R Z_C}{Z_R + Z_C} = \frac{R \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{R}{1 + j\omega CR}$$

$$= \frac{R}{\sqrt{1 + \omega^2 C^2 R^2}} \angle \tan^{-1}(-\omega C R)$$

4. Power in Sinusoidal Steady State

When the through or across variable applied to a linear system is sinusoidal, the instantaneous power flow to the system will, in general, be time-varying. However, we are often more interested in the average value of the power flow than in its instantaneous value. Systems for which this is true include electric power distribution systems, power radiated or received by communication antennas, power transmitted by ultrasound devices, etc. For example, the voltage supplied to your house by the electric company varies sinusoidally at a rate of 60 cycles per second. If this voltage is applied to, say, a lamp, the heat and light produced also vary periodically in amplitude (but at 120 cycles per second)' To determine the amount of energy consumed (for example, to calculate the electric bill), the instantaneous values of power during the 1/120 second is not as important as the average power consumed. A convenient time period for calculation of average power is one period of the wave since everything repeats itself every cycle.

2. Power calculations in steady state sinusoidal electrical systems

The analysis in Part 1 of this Chapter assumed that the AC signal was applied to a pure dissipative element (a resistor). In this case, the familiar forms for power, I^2R and V^2/R could be used as long as we first converted i(t) or v(t) to effective values. However, when AC signals are applied to impedances, we must modify this approach. Consider the general case of Figure 3. An AC voltage source drives a system which contains a combination of Rs, Ls, and Cs. In power computations, we often refer to the system as the "load" on the power supply.

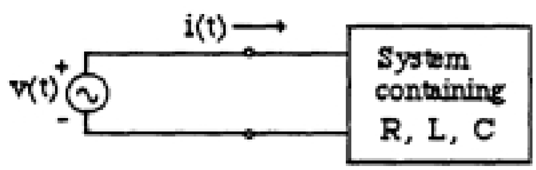


Figure 3
AC power delivered to a load

Assume the voltage source is $V_m \cos(\omega t)$. The current drawn by the load will in general not be in phase with this voltage since the load is not a pure resistance. We will always use the symbol θ to designate the angle of the voltage minus the angle of the current. Note that this is the angle of the impedance:

$$Z = \frac{V}{I} = \frac{V\angle\alpha}{I\angle\beta} = |Z|\angle(\alpha-\beta) = |Z|\angle\theta$$

$$v(t) = V_m \cos(\omega t) \qquad\qquad i(t) = I_m \cos(\omega t - \theta) \qquad (11)$$

The product of v(t) and i(t) is instantaneous power, p(t).

$$p(t) = v(t)\, i(t) = V_m\, I_m \cos(\omega t) \cos(\omega t - \theta). \qquad (12)$$

identity The cosine product on the right . side of equation 12 can be simplified by use of the trig

$$\cos(A)\cos(B)=1/2\,[\cos(A+B)+\cos(A-B)]$$

where we take ωt as A and θ as B. After substitution into Equation 12, the result for instantaneous power is

$$p(t) = \frac{V_m I_m}{2}\left[\cos(2\omega t + \theta) + \cos(\theta)\right] \qquad (13)$$

A plot of p(t) is given in Figure 4. Note that the first term of Equation 13 is a sinusoid with frequency twice that of the voltage or current' This is consistent with the sinusoidal waves of Figures 1 and 2. Because it is a sinusoid, the first term of Equation 13 has an average value of zero. The second term of Equation 13 is a constant. The average power, P, is then

$$P = \text{ave}(p(t)) = \frac{V_m I_m}{2}\cos(\theta) = V_{eff}\, I_{eff}\cos(\theta) \qquad (14)$$

where the subscript eff stands for "effective value". Effective value is the square root of the average value of the square of the function. For sinusoids, this is the maximum value divided by fT. All references to voltage or current found on AC equipment nameplates (e.g., 120V on a light bulb or 6.OA on a motor) refer to effective value unless otherwise specified. In other words, if your house's electric system is said to supply 120 volts AC, it really provides something like 170 cos(377t) volts' <u>In this sinusoidal steady state section of our course, any capital V or I without a subscript will be understood to stand for effective value.</u>

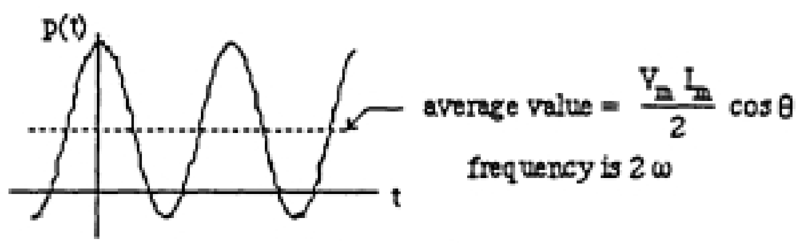


Figure 4
Instantaneous power, p(t) delivered to the system of Figure 3.

Note that the power flow oscillates and power is actually returned by the load to the source during portions of the cycle (negative regions). The average power to the load depends not only on the effective values of voltage and current, but also on the cosine of the phase angle between them. For example, if the load is a pure resistance, the phase angle is zero, the cosine is 1, and the average power is P = Veff Ieff. On the other hand, if the load is a pure reactance (contains Ls and/or Cs, but no Rs), the phase angle is ±90°, the cosine is zero, and the average power flow is zero. These cases are illustrated in Figure 5.

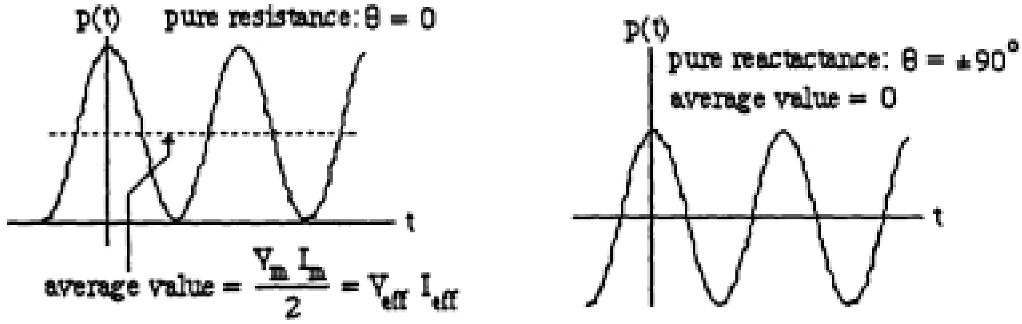


Figure 5
Left: pure resistance load. Power oscillates, but flow is always positive' P = Veff Ieff.
Right: pure reactance load Power flows to storage in the load, and then back to source. P = 0.

Power Factor

The cosine of the phase angle between voltage and current (the angle of the impedance) is given the name "power factor", abbreviated pf. A unity power factor means the load is a pure resistance' A zero power factor indicates a purely reactive load (contains only L and C). From Equation 14, the average power in terms of power factor is:

$$P = I_{eff} V_{eff}(pf)$$

$$pf = \cos\theta \tag{15}$$

. Therefore power factor is a positive number between 0 and 1 which indicates how much of the "apparent power", Veff ff Ieff, actually is dissipated by the load. Power factor is sometimes expressed as a percentage.

Example 3. Given $v(t) = 150 \cos(\omega t + 10^o)$,
$i(t) = 5 \cos(\omega t - 50^o)$.
Find average power, P.

Solution: Current lags voltage by $10^o - (-50^o) = 60^o$, therefore $pf = \cos(60^o)$.
From Equation 14,

$$P = \frac{(150)(5)}{2} \cos 60^\circ = 187.5 \text{ watts}$$

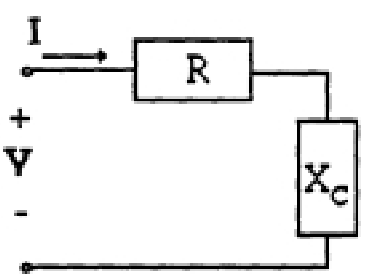
Example 4. When connected to a voltage source v(t) = 99 cos(6000t + 300), a certain load, made of two ideal series elements, dissipates an average power of P = 940 watts. The power factor of the load is known to be 0.707 with current leading the voltage. Find the two series elements.

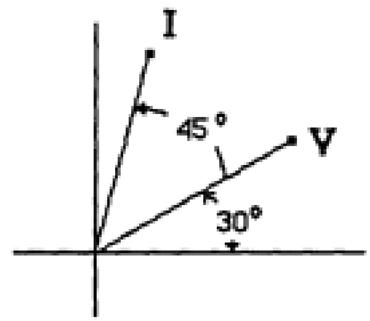
Solution: Use Equations 14-15: $P = V_{eff} I_{eff} (pf)$

$$V_{eff} = \frac{99}{\sqrt{2}} = 70$$

$$940 = 70\, I_{eff} (.707) \quad \Rightarrow \quad I_{eff} = 19.0$$

Since the power factor is neither zero nor unity, one of these ideal elements must be a pure resistance, and the other a pure reactance (L or C). The leading current tells us that this is an R-C combination. We are given that the phase position of the voltage phasor is 30°. The '707 leading power factor (cosine of the angle between current and voltage) means that the angle of the current phasor is at an additional 450 or (300 + 45°) = 75°.





Since the two elements are in series, they both get the same current: Ieff = 19.0. As this current flows through the resistance, it dissipates 940 watts (the average power flow to the capacitor must be zero). So a resistor with an effective current of 19 amps is dissipating 940 watts:

$$P = (I_{eff})^2 R \qquad 940 = 19^2 R \quad \Rightarrow \quad R = 2.6\Omega$$

The value of C can be determined in two ways: we can use the phase angle information or the impedance magnitude information. The impedance of the series combination is

$$Z = R + \frac{1}{j\omega C} \; . \quad \text{The phase angle of Z is } \tan^{-1}\left[\frac{\frac{1}{\omega C}}{R}\right]. \quad \text{Magnitude is } \sqrt{R^2 + (\frac{1}{\omega C})^2}$$

We know the power factor is .707 which is the cosine of the impedance angle, θ. Therefore this angle is ±45°. Its sign must be negative because current leads voltage. Therefore C can be found from setting $-1/(\omega CR) = -0.707$ and solving for C ($\omega = 6000$).

A second approach is to use the fact that magnitude of Z is equal to magnitude of V divided by magnitude of I:

$$|Z| = \frac{|V|}{|I|} = \frac{70}{19} = \sqrt{R^2 + X_C^2}$$

By either method, C = 64.1 μF.

Average Power to Impedance or Admittance

We have found the following general formula for average power to any

$$P = V_{eff}\, I_{eff} \cos\theta. \qquad (16)$$

Frequently, we are given only V or I along with the load impedance, Z or admittance, Y. In these cases, the following formulas can be used to determine average power'

Multiply and divide the right side of Equation 16 by Ieff:

$$P = \left[\frac{V_{eff}}{I_{eff}}\right] I_{eff}^2 \cos\theta$$

The ratio of Ve ff to Ieff is IZI, the magnitude of the impedance. But the magnitude of a complex number times the cosine of its angle is the real part of the number. Therefore,

$$P = I_{eff}^2\, \mathcal{R}e(Z) \qquad (17)$$

A similar operation can lead to a formula for average power in terms of Veff. Multiply and divide the right side of Equation 16 by Veff.

$$P = \left[\frac{I_{eff}}{V_{eff}}\right] V_{eff}^2 \cos\theta$$

The ratio of Ieff to Veff is IYI, the magnitude of the admittance. Again, the magnitude of a

complex number times the cosine of its angle is the real part of the number (although the angle

$$P = V_{eff}^2\, \mathcal{R}e(Y) \qquad (18)$$

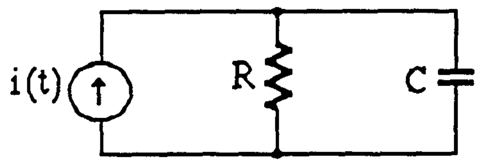
When calculating average Rower in AC systems it is very important always to start with one of these three Equations (16. 17. 18).

Example 5

Find the average power, P, delivered to the circuit below by the current source. The current source value is

$$i(t) = 5\cos(10^5 t + 53^{o})$$

The circuit elements are: $R = 5\Omega$, $C = 4\ \mu F$.



. Solution: Since current is given, we prepare to use Equation 17.

$$\mathbf{Y} = G + j\omega C = \frac{1}{5} + j\,(10^5)(\,4 \times 10^{-6}) = 0.2 + j\,0.4$$

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{1}{0.2 + j\,0.4} = 1 - j\,2, \qquad \mathrm{Re}(Z) = 1$$

$$I_{eff} = \frac{5}{\sqrt{2}}\,, \qquad P = I_{eff}^2\,\mathrm{Re}(Z) = \frac{25}{2}(1) = 12.5 \text{ watts}$$

3. Complex Power

An alternate way of computing power flow in steady state AC systems is to work with quantities which have the dimension volt-amperes instead of voltage and current separately. Consider the voltage and current phasors shown on the left panel of Figure 6. (All magnitudes on this diagram are assumed to be effective values.) The current lags the voltage by θ degrees. (Note that this means that θ, the angle of the impedance, is positive.) The current phasor can be separated into two components, one in phase with the voltage, and one 90° out of phase, as shown.

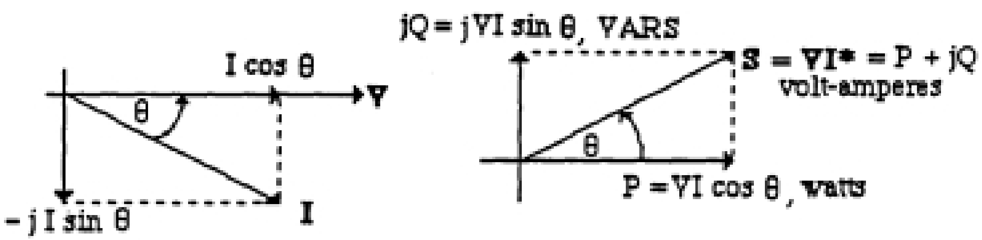


Figure 6
Left: Voltage-Current phasors.Right: Power Triangle

If we multiply each of the current phasors of the left panel by the magnitude of the effective voltage, and draw θ as a positive angle,we get the diagram on the right. The complex number $\mathbf{S}$, and its components P and Q, each have the dimension volt-amperes. However, we give special names to each of these dimensions. The VI cosθ component is recognized as average power to the load, P, and its unit is therefore the watt. The jVI sinθ component represents the average power flow to or from the reactive part of the load during each quarter cycle. (This flow reverses during alternate quarter cycles.) It is represented by the letter Q and its unit is volt-amperes reactive, or VARS. The resultant, **S**, called "complex power", is a complex number of magnitude VI. (The name "apparent power" is sometimes used for the VI product.) Its angle, θ, is the angle of the impedance. $S = P + jQ$

By definition, power, P, must be measured on the real axis, and Q must be measured on the imaginary axis. Since the phasors **V** and **I** may be at any angular position, **S** is not simply the complex product of the **V** and **I** phasors. (For example, if **V** and **I** are in phase, both at 20°, the **VI** product will add the angles to give 40° instead of zero.) Assume **V** is $V\angle\alpha$, and **I** is $I\angle\beta$. The angle of the impedance is then $\theta = (\alpha-\beta)$. This angle is obtained if we multiply the voltage phasor by the complex <u>conjugate</u> of the current phasor.

Some important relations:

$$\mathbf{S} = \mathbf{VI}^* = I^2\mathbf{Z} = P + jQ$$

$$P = VI\cos\theta = I^2\,\mathrm{Re}(\mathbf{Z}) = V^2\,\mathrm{Re}(\mathbf{Y}), \quad \text{watts}$$

$$Q = VI\sin\theta = I^2\,\mathrm{Im}(\mathbf{Z}) = -V^2\,\mathrm{Im}(\mathbf{Y}), \quad \text{VARS}$$

$$(\theta \text{ is angle of } \mathbf{Z}, \text{ the negative of the angle of } \mathbf{Y}) \tag{19}$$

Power Factor Correction

The usual electric power system has multiple loads connected across the same voltage. For example, an industrial plant may have lighting loads, machinery loads, heating system loads, etc., all connected across the same line voltage' Resistive loads such as lighting or heating, generally draw close to unity power factor current. Transformers and heavy machinery generally draw a lagging current.

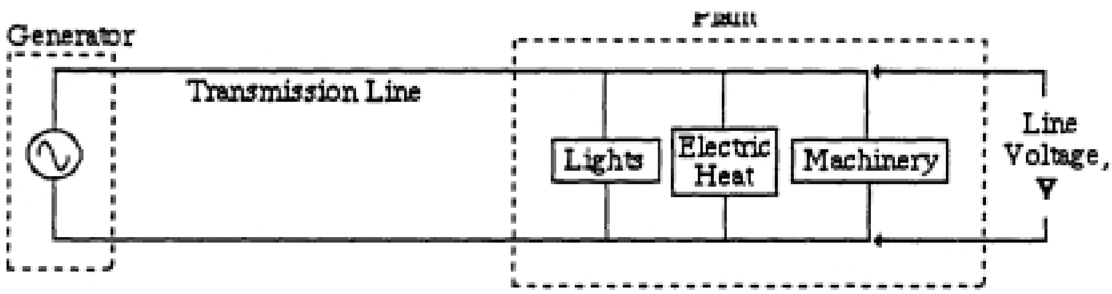


Figure 7

The electric company charges only for energy consumed, which means the integral of P dt. We know that $P = V_{eff}I_{eff}\cos\theta$, and that $(I_{eff}\cos\theta)$ is the component of the total effective current which is in phase with the voltage. What about $I_{eff}\sin\theta$, the 90° out-of-phase component of exchange taking place between the source and the reactive components of the various loads. Theoretivcally, no power is consumed by this action and therefore no fuel needs to be burned by the electric company to turn the generator to produce the 90° out-of-phase component of current. Practically, however, this is not the case. The transmission lines carrying current from the generator to the load must carry &U the current, not just the in-phase component. Until someone invents a resistanceless transmission line, 12R losses will occur in transmission wires and will be a function of the total current.

Since I = VYL = V(GL+jBL), a reduction in BD the susceptive part of the load admittance, will reduce the transmission line current magnitude, I, without affecting the average power to the load (V2GL). To avoid excessive transmission line losses, power companies usually require industrial loads to maintain a power factor greater than some specified value, e.g. 80%.

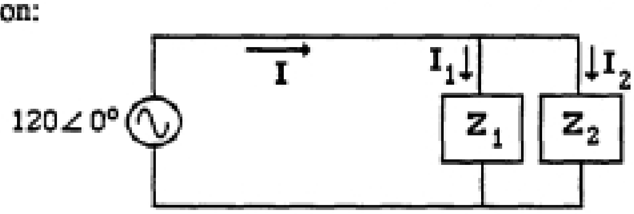
Example 6

Two pieces of heavy machinery are connected across a 120 volt, 60 Hz line. Their impedances are $\mathbf{Z}_1 = 4\angle 30^\circ$ and $\mathbf{Z}_2 = 5\angle 60^\circ$.

(a) Determine the average power to the load and the total line current.

(b) Reduce the line current by adding an appropriate parallel element.

Solution:



$\mathbf{Z}_1 = 4\angle 30^\circ = 3.464 + j2$

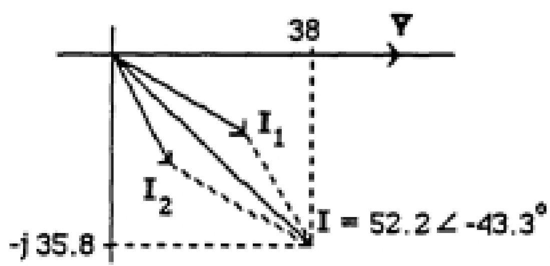
$\mathbf{Z}_2 = 5\angle 60^\circ = 2.5 + j4.33$

$$\mathbf{I}_1 = \frac{120\angle 0^\circ}{4\angle 30^\circ} = 30\angle{-30^\circ} = 26 - j15 \qquad \mathbf{I}_2 = \frac{120\angle 0^\circ}{5\angle 60^\circ} = 24\angle{-60^\circ} = 12 - j20.8$$

$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 38 - j\,35.8 = 52.2\angle{-43.3^\circ}$ Total line current is <u>52.2 amps</u>.

Total volt amperes is (120)(52.2) = 6264.

pf = cos (-43.3°) = 0 7277



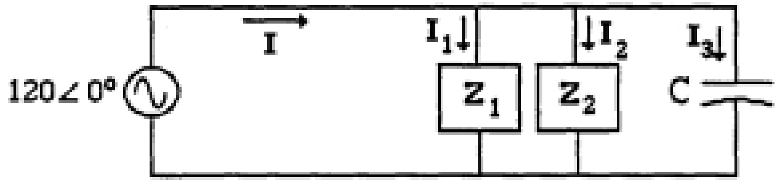
Power: $P_{Z1} = (120)(30)\cos(30^\circ) = 3118$ watts. (Also, $I^2\mathrm{Re}(\mathbf{Z}) = (30^2)(3.464) = 3118$)

$P_{Z2} = (120)(24)\cos(60^\circ) = 1440$ watts. (Also, $I^2\mathrm{Re}(\mathbf{Z}) = (24^2)(2.5) = 1440$)

Total Power = $P_1 + P_2 = 3118 + 1440 =$ <u>4558 watts</u>.

Check: VI pf = (120)(52.2)(0.7277) = 4559 watts.

Since the combined load draws a lagging current, add a capacitor in parallel with these loads across the line:

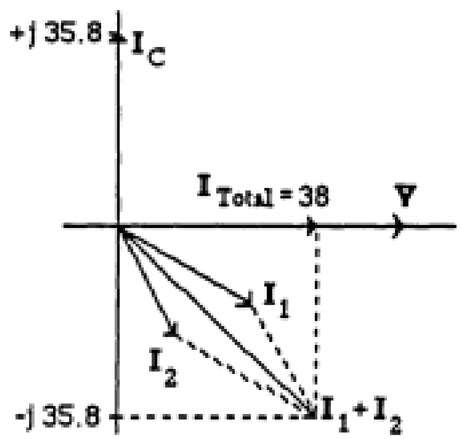


The capacitor should be such that it draws an additional 900 leading current which cancels the 35'8 amp 900 lagging current drawn by the parallel combination of Z1 and Z2' The addition of the parallel canacitor does not affect the currents in Z, or Z.,. The impedance of the capacitor. Z,.

= 1/jωC must equal the voltage to I_3 ratio:

$$\omega = 2\pi f = 2\pi(60) = 377$$

$$Z_C = \frac{120\angle 0°}{35.8\angle 90°} = 3.35\angle -90° = \frac{1}{j(377)C} \qquad \text{or} \quad C = 791\ \mu F$$



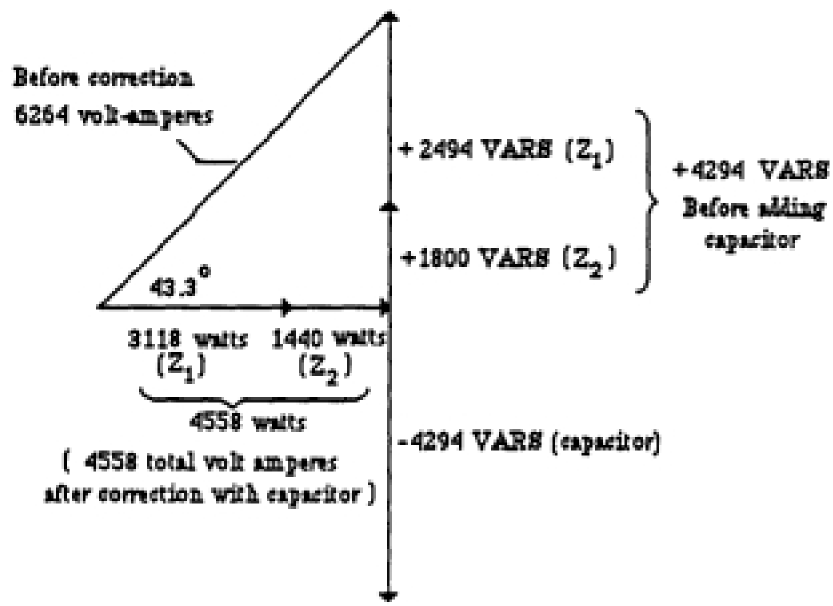
Note that by adding an additional parallel element, we have reduced the total line current from the previous 52.2 amp ($|I_1 + I_2|$) to 38 amp ($|I_1 + I_2 + I_C|$). However, both Z_1 and Z_2 still receive the same current and power as before. To get an idea of the possible reduction of losses this provides, assume the transmission line has a total equivalent resistance of 0.1Ω. The original system drew a total current of 52.2 amp, so the line loss was $I^2R = 272.5$ watts. With the addition of the capacitor, the total line current is 38 amp, and the loss is reduced to 144.4 watts. Thus the addition of the parallel capacitor has achieved a 47% reduction in line power losses.

This example could also have been done using the concept of complex power. The values of P and Q drawn by the original two loads can be found using Equation 19:

$P = VI\cos\theta$	$Q = VI\sin\theta$	(θ is <u>impedance</u> angle)
$P_1 = (120)(30)(\cos 30^o) = 3118$ watts,	$Q_1 = (120)(30)(\sin 30^o)$	$= 1800$ VARS
$P_2 = (120)(24)(\cos 60^o) = \underline{1440 \text{ watts}}$,	$Q_1 = (120)(24)(\sin 60^o)$	$= \underline{2494 \text{ VARS}}$
Totals: 4558 watts		4294 VARS

To reduce the reactive power to zero, we need to draw an additional -4294 VARS. For negative Q (negative impedance phase angle), we need a capacitor. Using Equation 19,

$$-4294 = Q_C = -V^2\,\text{Im}(Y_C) = -(120^2)(377C), \text{ from which } C = 791\mu F.$$



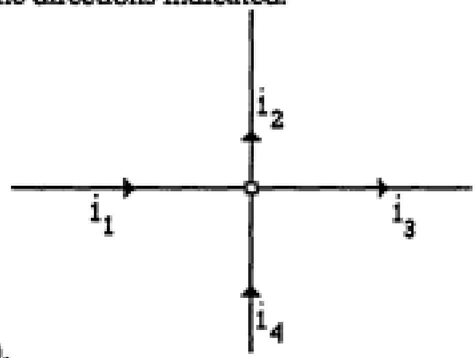
1. At the node shown, currents are:

$i_1 = 13\cos(10t - 67.38^{o})$

$i_2 = 5\cos(10t + 36.87^{o})$

$i_3 = 9\cos 10t.$

in the directions indicated.



Find $i_4(t)$.

2. Three of the currents flowing into the node (shown below) are:

$I_1 = -10\cos 2t$

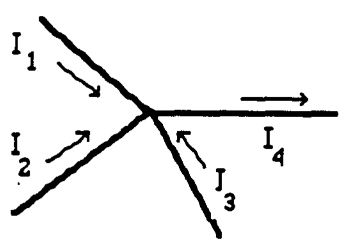
$I_2 = 10\cos(2t + 30^{o})$

$I_3 = 5\cos(2t - 130^{o})$

(a) Express each as a complex phasor.

(b) Express I_4 in the form $A\cos(\omega t + \Phi)$.

(give values for A, ω, and Φ)



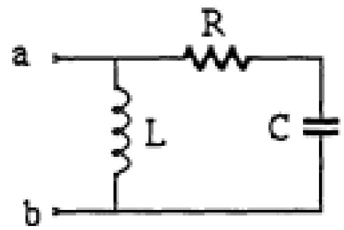
3. Given $i_1 = 13\cos(10t - 67.38^{o})$, $i_2 = 5\cos(10t + 36.87^{o})$, and $i_3 = 9\cos 10t$. Find $i_1 + i_2 + i_3$.

4. The load below is driven by a steady state sinusoidal current source of frequency $\omega = 5000$ rad/sec $R = 13\Omega$, $L = 26$ mH, and $C = 7.69\ \mu F$

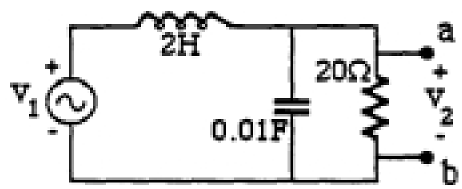
(a) Find the complex impedance looking into terminals a-b.

(b) Find a parallel two-element circuit equivalent to this one at this frequency between terminals a-b (sketch the equivalent circuit and specify the element values).

(c) If the peak current from the source is 1 amp, find the average power dissipation of this system.



5. For the circuit below, the source is v1(t) = 20 cos(5t + 20°). Find v2(t)' Show both V 1 and V2 on a phasor diagram.

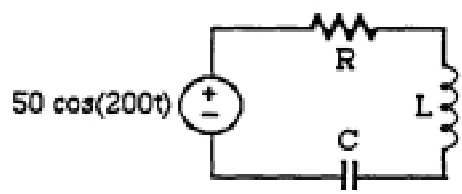


6. For the circuit shown below, $R = 3\Omega$, $L = 30$ mH, $C = 2500\ \mu F$:

(a) Find the current, i(t). (answer: $i = 10\cos(200t\text{-}53^{o})$)

(b) Find the phasor voltage across each element (R, L, C), and sketch each on a phasor diagram.

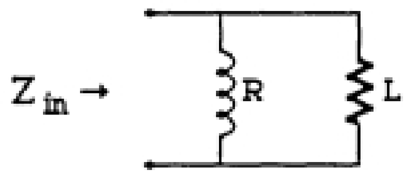
(c) Express each voltage found in (b) as a function of time.



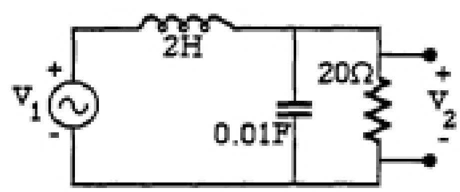
7. (a) Find the input resistance Rin = (Re(Z)) and input reactance Xin = (Im(Z)) for the system shown.

(b) Find the input conductance Gin = (Re(Y)) and input susceptance Bin = (Im(Y)) for the same system.

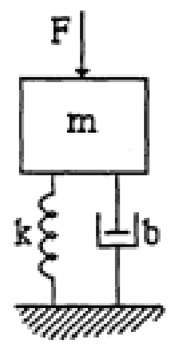
(c) What additional parallel element (type and value) will make the system look like a pure resistance at the input terminals??



8. For the circuit below, the source is v1(t) = 20 cos(5t + 20°). Find v2(t). Show both voltages on a phasor diagram.



9. You want to annoy some people in a parked car (its mass and suspension system can be represented by the model shown below), so you start pushing up and down (sinusoidally) on its bumper. The mass of the car is 1800 kg. Because of a broken shock absorber, b is negligible. The car's spring suspension system has k = 71000 n/m. = At what frequency (in pushes per second) should you push so that the occupants experience the greatest annoyance?



10. A 220 volt induction motor delivers a 2 HP output and operates at 85% efficiency. Power factor is 0.8, lagging.

(a) Determine the line current. (Ans. 9.95L-36.9°)

(b) What capacitor added across the line (in parallel with the motor) will correct the power factor to unity? (Ans. 72µF).

Systems Chapter 16 Study Guide

Periodic Functions and Fourier Series

A. **Concepts Addressed By This Chapter**

1. Periodic Function
2. Average and Effective Values
3. Fourier Series
   - Trigonometric Form
   - Exponential Form
4. Applications to System Solutions.

B. Introduction

The steady-state solution of system behavior is often required when the forcing functions (and therefore the forced response terms) are periodic. That is, waveforms which repeat at regular intervals. In many cases, the actual waveform is not as important as is the power or enegy it delivers over a period of time. Since power is a function of v2 or f2, it is not constant for periodic waves. However, the average value of power during each cycle of a periodic wave is a more useful quantity, and is a constant. The effective value of the across or through variable can be used to determine this average power.

One periodic waveform we already know how to handle is the sinusoid. We have studied techniques for determining the response of a linear system to constant or sinusoidal forcing functions. However, if x(t) is periodic but not sinusoidal, the response to this input may be difficult to find directly. The Fourier series allows x(t) to be approximated by a (finite) number of sinusoidal terms of different frequencies. The response of a system to each of these terms may be found separately using sinusoidal analysis. Then, by superposition, these responses can be added to approximate the system response to a general periodic x(t).

C. Instructional Objectives

A student mastering this material will be able to

1. Determine the period and frequency of periodic waves.
2. Find average and effective values of periodic waves.
3. Given a graph or an expression for a periodic wave, determine the coefficients of the trigonometric or exponential Fourier series approximation for the wave.
4. Sketch magnitude and phase spectrum plots of Fourier series terns.

D. Study Procedure

Read Chapter 16 Additional material can be found in references 1 and 13'

Chapter 16
Periodic Functions and Fourier Series

1. Periodic functions:

A periodic function is one which exactly repeats itself every T units along the axis. Mathematically, this can be expressed

$$x(t) = x(t + T) \qquad (1)$$

where t is time and T is the *period* of the wave. Examples of some periodic functions are given in Figure 1. For each case the period is designated by T.

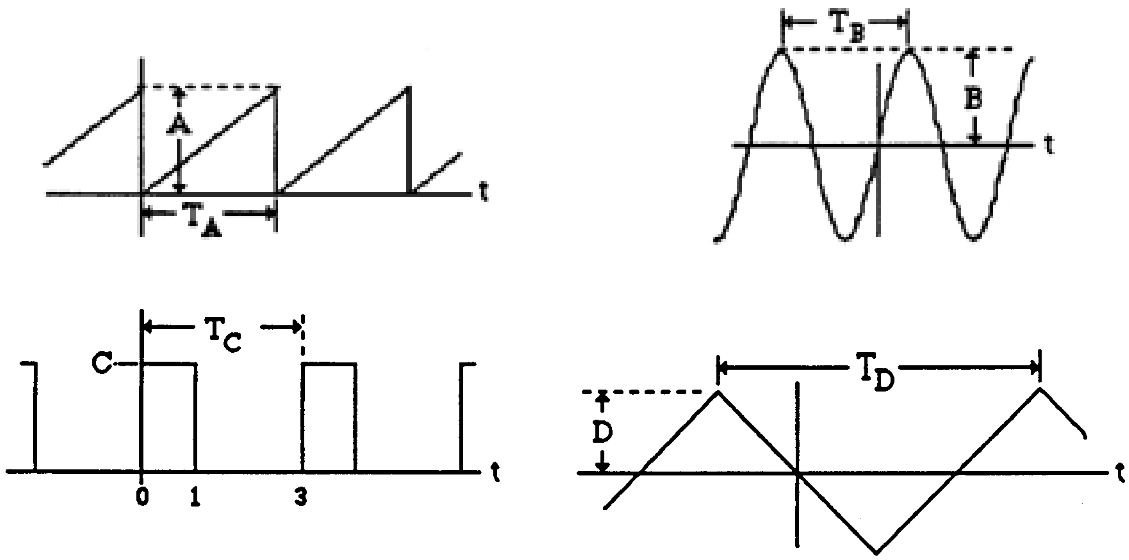


Figure 1
Periodic waves. Period = T

It may seem from Figure 1 that periodicity is easy to observe and that the value of the period, T, can readily be obtained from the graph of the function, or from its mathematical expresion. This is not always the case for combinations of periodic waves. Consider the sum of any two periodic waves with periods T1 and T2. We observe the two waves (waves 1 and 2) and their sum (wave 3) at a given starting point on the time axis, and move along the time axis to find a point where all the same values repeat. Suppose T1 = 2 and T1 = 4. Then, wave 1 begins to repeat after 2s, but wave 2 is only halfway through its period at that time. After 4s, wave 1 again begins to repeat, and now so does wave 2. Therefore, the period of wave 3 is T2. What if t1 = 2 and T2 = 3? Now 6s are required before both waves are back at their original values together. So in this case T3 = 6s. A rule we can establish from this is that the period of the sum of several periodic waves is the lowest common multiple of the individual wave periods' What if one wave has period = t7 and the other has period 4? Then their sum is not periodic because the two never return to their starting value together.

1.1 Power Associated with a periodic function

Suppose any of the waves of Figure 1 is a current flowing in a 1Ω resistor. The average values of these waves (average value can be found by dividing the area under the curve over one cycle by T) are A/2, 0, C/3, and 0, respectively. However, average power cannot be determined from the average value of the current.

Power to the resistor is i2R or v2/R, so power flow vs. time in each case will also be periodic (but not necessarily the same period as the current or voltage). Figure 2 shows each of the waves of Figure 1 after squaring values at each point in time. If the waves of Figure 1 are currents, then the waves of Figure 2 have dimension current squared and represent the power those currents would dissipate in a W resistor. Note the differences in shape and period which often appear when we compare a plot of either the across or through variable and a plot of power flow in sinusoidal steady state systems.

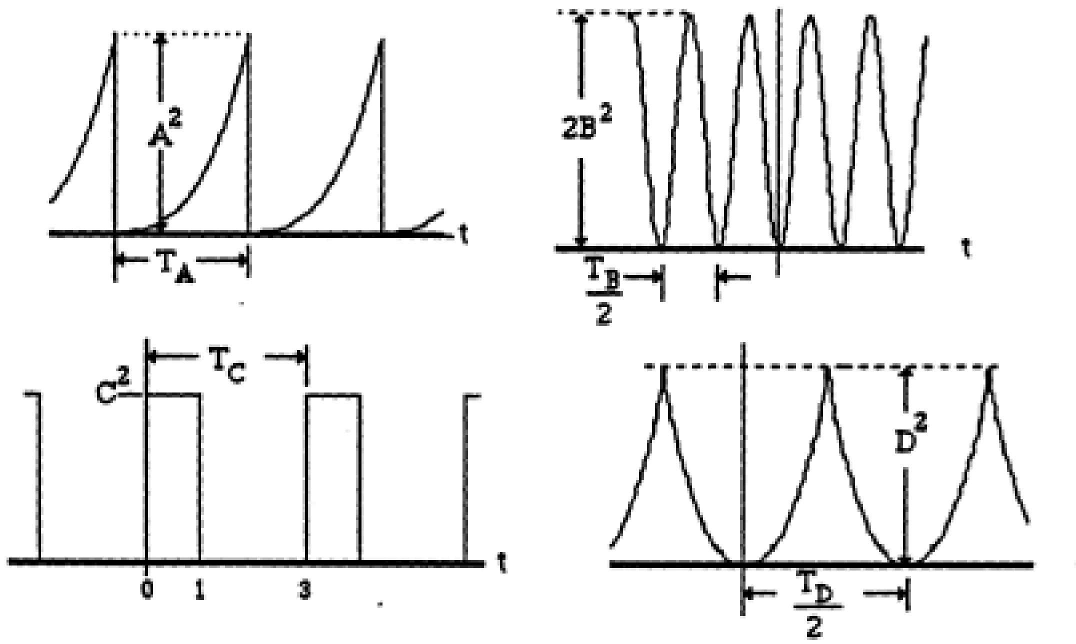


Figure 2
Power each of the currents of Figure 1 would produce in a 1Ω resistor.

The plots of Figure 2 provide values of instantaneous power, p(t). In many cases, (e.g. electric appliance, light bulb, motor) <u>average</u> power flow is of greater interest. To calculate average power, P, we need to find the average values of waves such as given in Figure 2. That is,

$$P = \text{average power} = \text{ave}\,[p(t)] = \text{ave}\left[\frac{v^2}{R}\right] = \frac{1}{R}\,\text{ave}\,[v^2]$$

$$\text{or} \qquad P = \text{ave}\left[i^2 R\right] = R\,\text{ave}\,[i^2] \qquad (2)$$

So, to calculate average power delivered to a D-type element under steady-state sinusoidal conditions, we need to find the average value of the <u>square</u> , of either the across or associated with the D-type element.

1.2 Effective Value

We would like to preserve the $p = i^2R$ and $p = v^2/R$ relations we have developed for instantaneous power dissipation. If i and v are periodic, average power is given by Equation 2. Therefore, define I_{eff} or V_{eff} (the "effective" value of i or v), as

$$I_{eff} = \sqrt{\text{ave}\,[i^2]} \quad , \quad V_{eff} = \sqrt{\text{ave}\,[v^2]} \tag{3}$$

Then the formula for average power dissipation in a D-type element, R, in a sinusoidal steady state system is:

$$P = I_{eff}^2 R \qquad \text{or} \qquad P = \frac{V_{eff}^2}{R} \tag{4}$$

From Equation 3, we see that the effective value is the square root of the mean (average) value of the squared function. For this reason, the letters rms (root-mean-squared) are sometimes used in place of *eff.* To find the effective value of a given periodic wave, it is necessary to square the function of time, find the average value of the squared function over one period, then take the square root of the result. Sometimes this can be done geometrically. For example, consider wave C of Figure 1. For numerical computation, assume the amplitude, C = 4. Squaring the function gives wave C of Figure 2. The average value of this wave is the area under one cycle, divided by TC. The area is 16 so the average value of the squared curve is 16/3 or 5.333. The effective value is the square root of this or 2.31. Therefore, if wave C of Figure 1 is a current supplied to a 291 resistor, the average power dissipation in the resistor would be P = 4.62 watts.

When waveforms are not easily handled by geometry, Equation 5 can be used to calculate effective value. The integral can be taken over any full period, T.

$$V_{eff} = \sqrt{\frac{1}{T}\int_T v^2\,dt} \qquad I_{eff} = \sqrt{\frac{1}{T}\int_T i^2\,dt} \tag{5}$$

<u>Example 1: Effective value of a sinusoid</u>

The primary subject of this unit is the analysis of systems driven by sinusoidal waves. Therefore, we will now determine the effective value of a sinusoid using Equation 5. Since effective value is independent of the phase position of the wave, we will choose a pure cosine function, $v(t) = V_m \cos(\omega t)$ as our time function. The period of this wave is T where $T = 2\pi/\omega$. Then, from Equation 5,

$$V_{eff}^2 = \frac{V_m^2}{T}\int_0^T \cos^2 \omega t\,dt \tag{6}$$

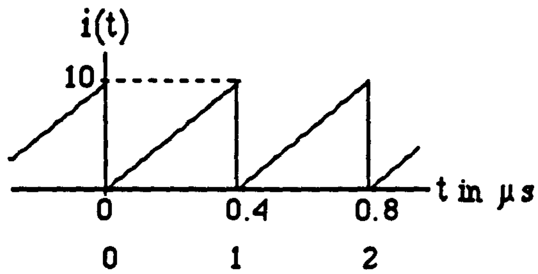
I

To integrate $\cos^2$, use the identity $\cos^2\omega t = (1/2) + (1/2)\cos 2\omega t$. After integration and substitution of limits, the first term yields T/2. The second term integrates to zero since the area o a cosine function over any number of full periods is zero. Therefore,

$$V_{eff}^2 = \frac{V_m^2}{T}\,\frac{T}{2} = \frac{V_m^2}{2} \tag{7}$$

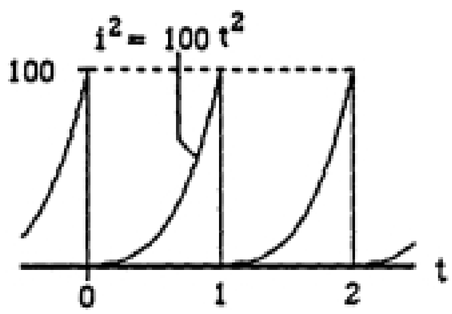
The effective value of a sinusoid is its peak value divided by $\sqrt{2}$.

<u>Example 2</u> Find the average power dissipated in a 5Ω resistor for the periodic current waveform given below:



This wave repeats every 0.4μs ($T=0.4\times10^{-6}$),which is a somewhat inconvenient time scale. For example, to square and then integrate i(t), we need a mathematical expression for i(t) over one period. In this case $i(t) = 2.5\times10^6 t$, so i^2 will be $6.25\times10^{12}t$. A simplification can be achieved if we realize that <u>the average value of a periodic wave is independent of the period</u>. Therefore, for the purpose of determining average values, we can choose any convenient scale for the time axis. The most convenient is the 0, 1, 2 scale printed under the original scale. The function during the first period is now $i(t) = 10t$.

According to Equation 5, we must square this function. The result is given below:

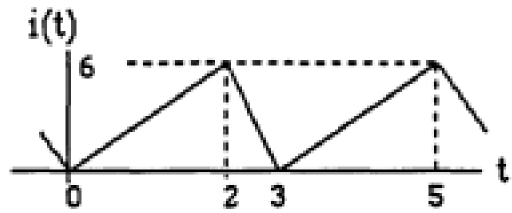


The squared function over the first period is $100t^2$. The integral of this evaluated from 0 t 1 is 100/3. Therefore, the effective value is $10/\sqrt{3}$. The average power this current dissipates in a 5Ω resistor is $(I_{eff})^2 R$ or 166.67 watts.

Note: Parabolic shapes like the one above will result whenever a ramp current or voltage function appears. The area under such a parabola is (1/3) bh where b is the base of the parabola and h is the height. Therefore, it should not be necessary to actually carry out the integration in most cases of ramp-type waveforms. See Example 2, below.

Example 2. Find the effective value of the triangular wave i(t) shown below.

Solution: Separate one cycle of the wave into its separate ramp segments (t=O to 2 and t--2 to 3). Squaring each ramp results in parabolic shapes of peak value 36. The effective value of i(t) is the square root of the sum of the areas under these parabolic segments divided by T'



$$I_{eff} = \sqrt{\frac{\text{area under } i^2}{T}} = \sqrt{\frac{\frac{1}{3}(2)(36) + \frac{1}{3}(1)(36)}{3}} = \sqrt{12}$$

where the area under each parabola was calculated using 1/3 its base times its height.

1.3 Effective value of a sum of sinusoids of different frequencies

Superposition does not apply for power since power is not a linear function of the across or through variable. Moreover, when periodic waves are added, the result may not even be periodic. (The sum is periodic if ratio of the periods of the added waves is a rational fraction which means a fraction in which numerator and denominator are both integers). We therefore should not expect the effective value of a sum of periodic functions to be equal to the sum of the individual effective values. In general, the waves must first be added and then Equation 5 may be applied to the sum. A very common special case is the sum of sinusoids of different frequencies. Consider a series of sinusoids in which all frequencies are integer multiples of a given frequency, ω_1. ω_1 may be called the *fundamental* frequency and its multiples called *harmonics*. Such a series has the name Fourier series and will be the subject of a later study. The series is expressed by Equation 8. It is periodic with a period equal to that of the slowest component, term 1 (the fundamental). The period of this term is $2\pi/\omega_1$ which we will call T_1.

$$i(t) = I_{1m}\cos(\omega_1 t+\phi_1) + I_{2m}\cos(2\omega_1 t+\phi_2) + I_{3m}\cos(3\omega_1 t+\phi_3) + \cdots \tag{8}$$

To find the effective value of this current, we must square i(t), then integrate it over one cycle. When we square the right side of Equation 8, we get the square of each term, plus 2 times the product of each pair of terms. The product terns will integrate to zero because of an identity given in Equation 9 (you may want to verify this). It states that zero results when we integrate two sinusoids of different frequencies (m and n are integers) over a complete period (T may be set to 2n).

$$\int_T \cos(mx)\cos(nx)\,dx = 0 \qquad (9)$$

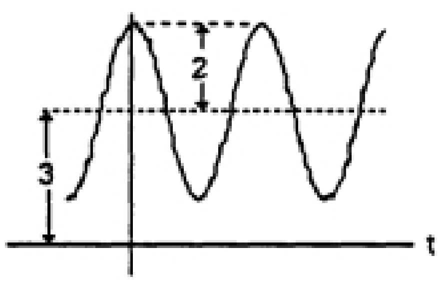
Therefore, squaring the right side of Equation 8 and integrating over a period produces the integral of the sum of the squares of the individual terms of Equation 8. But we have already seen (Equation 6) that these integrals are simply the square of the effective value of each sinusoid.in the series. Therefore, for the case of a Fourier series given by Equation 8, the effective value of the series is the square root of the sum of the squares of the individual components. This is expressed in Equation (10).

$$I_{eff} = \sqrt{I_{eff1}^2 + I_{eff2}^2 + I_{eff3}^2 + \dots} \qquad (10)$$

Because any periodic wave can be represented by a Fourier series, Equation 10 can also be applied in cases of non-sinusoidal summations.

Example

The waveform below is a voltage applied to a 2Ω resistor. What average power will be dissipated?



The wave is $v(t) = 3 + 2\cos(\omega t)$. The effective values of the terms are 3 and $2/\sqrt{2}$, respectively. Therefore, using Equation 10, the effective value of the sum is $\sqrt{(9 + 2)} = 3.32$. The power this voltage would dissipate in a 2Ω resistor is 5.5 watts.

$x(t)$ is a *periodic* function of t if $x(t) = x(t + T)$, where T, a constant, is the *period*. That is, the wave repeats itself every T seconds along the time axis. You are already familiar with several periodic waves such as sinusoids, square waves, triangular waves, etc.

2. Fourier Series

Most periodic functions can be closely approximated by a finite number of terms from a series known as a Fourier series. The Fourier series contains a constant term equal to the average value of the given wave, and and a number (possibly an infinite number) of sinusoidal or exponential periodic terms.

2.1 Trigonometric Fourier series

$$x(t) = a_0 + a_1\cos\omega_1 t + a_2\cos 2\omega_1 t + a_3\cos 3\omega_1 t + \dots$$
$$+ b_1\sin\omega_1 t + b_2\sin 2\omega_1 t + b_3\sin 3\omega_1 t + \dots$$

or, $$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_1 t + \sum_{n=1}^{\infty} b_n \sin n\omega_1 t}$$ (11)

The a's and b's are constants. The lowest frequency periodic term (n = 1) is called the *fundamental* component and ω_1 is the fundamental frequency. Other periodic terms have frequencies which are integral multiples of the fundamental frequency, called *harmonics*. The relationship between T and ω_1 is

$$T = \frac{2\pi}{\omega_1} = \frac{1}{f_1} \qquad (12)$$

Suppose we have a periodic signal x(t) in the form of a graph or a mathematical expression and we wish to approximate it by a sinusoidal Fourier series. We can easily determine ω_1 from the period of the given wave, $\omega_1 = 2\pi/T$. What remains is to determine the proper a and b coefficients of the series of Equation 11. The formulas below (Equations 13a and 13b) permit evaluation of the coefficients from the given x(t). You may want to verify these formulas by hand or using Maple. The process is: To find any coefficient, say a_2, multiply both sides of Equation 11 by cos $2\omega t$, then integrate all terms over one full (fundamental) period. After multiplication, the terms on the right will consist of a $\cos^2(2\omega t)$ term, terms of the form $(\cos m\omega t)(\cos n\omega t)$ where $m \neq n$, and terms of the form $(\sin m\omega t)(\cos n\omega t)$. The integral over a full period of all but the $\cos^2(2\omega t)$ term will go to zero. The integral of $a_2 \cos^2(2\omega t)$ over the range T is $a_2\pi$. On the left we have the integral of $x(t) \cos 2\omega t$, allowing a_2 to be evaluated. Similar steps for the other harmonics result in the following formulas for the a_i and b_i coefficients:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{T}\int_T x(t)\,dt \\ a_n &= \frac{2}{T}\int_T x(t)\cos(n\omega_1 t)\,dt \\ b_n &= \frac{2}{T}\int_T x(t)\sin(n\omega_1 t)\,dt \end{aligned}}$$ (13a)

The T under the integral signs in Equations 13a means the integral covers one full period of x(t). It does not matter whether this integral is from 0 to T, or -T/2 to + T/2, or any other range, as long as it is over one complete period. Note that the a0 term is simply the average value of the periodic wave, that is, the area under the wave for one period, divided by the time for one period. Frequently it is easier to obtain the average value by inspection and possibly some geometry, than by application of the Equation 13a formula. For example, the average values of the four waves of

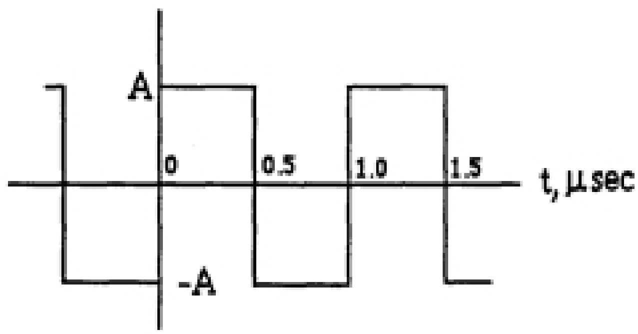
Figure 1 can be seen by inspection to be A/2, 0, C/3, and 0, respectively.

A second form for the formulas of Equations 13a can be obtained by changing the time axis to the $\theta = \omega_1 t$ axis. This is often helpful when the numerical values of the time variable are inconvenient. By using $d\theta = \omega_1\, dt$, and $\omega_1 T = 2\pi$, we can convert Equations 13a to Equations 13b:

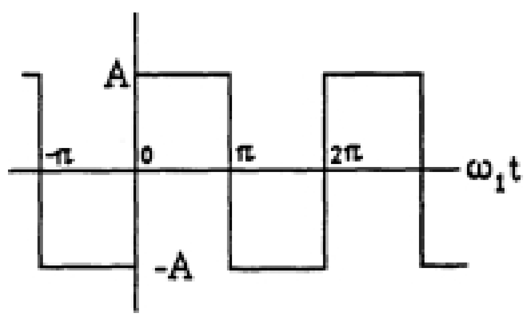
$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{2\pi} x(\theta)\, d\theta \\ a_n &= \frac{2}{\pi}\int_{2\pi} x(\theta)\cos(n\theta)\, d\theta \\ b_n &= \frac{2}{\pi}\int_{2\pi} x(\theta)\sin(n\theta)\, d\theta \end{aligned}}$$

Example:

Find the trigonometric Fourier series to approximate the 1 MHz square wave given



We see that $T = 1\ \mu\text{sec}$, and therefore $\omega_1 = 6.283\times10^6$. We could use Equations 13a to find the a_i and b_i values, but this would result in values such as 10^{-6} and 6.283×10^6 being distributed throughout the calculations, including the integration limits. A better aproach is to convert the t axis to the $\omega_1 t$ axis:



Now the integral for an from Equations 13b is

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(\theta)\cos(n\theta)\,d\theta = \frac{A}{\pi}\int_0^{\pi}\cos(n\theta)\,d\theta - \frac{A}{\pi}\int_{\pi}^{2\pi}\cos(n\theta)\,d\theta$$

The integral was was separated into two parts because the $x(\theta)$ function has two different values over the cycle, +A and -A. These integrals are simple enough to solve but in most cases, Maple provides an easier approach to integral evaluation. The Maple expression for the above integrals is:

(A/Pi)*(int(cos(n*x),x=0..Pi)-int(cos(n*x),x=Pi..2*Pi)); (here x was used instead of θ)

Using paper and pencil approach:

$$a_n = \frac{A}{n\pi}\sin n\theta\Big|_0^{\pi} - \frac{A}{n\pi}\sin n\theta\Big|_{\pi}^{2\pi} = 0$$

So all a coefficients (all cosine terms) are zero.

For bn, relace cos by sin in the above expressions:

$$b_n = \frac{A}{\pi}\int_0^{\pi}\sin(n\theta)\,d\theta - \frac{A}{\pi}\int_{\pi}^{2\pi}\sin(n\theta)\,d\theta = -\frac{A}{n\pi}\cos n\theta\Big|_0^{\pi} + \frac{A}{n\pi}\cos n\theta\Big|_{\pi}^{2\pi}$$

The result is: $$b_n = \begin{bmatrix} 0, \ n \text{ even} \\ \frac{4A}{n\pi}, \ n \text{ odd} \end{bmatrix}$$

So the series representation of the square wave is:

$$x(t) = \frac{4A}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

Note that $\omega_1 = 2\pi \times 10^6$ for the given data, but there is no need to insert this value during the calculation of the coefficients. This series contains an infinite number of terms. The more terms we add together, the more the sum looks like the square wave. How many terms must we calculate? It depends on how much error is tolerable. A rule of thumb might be to include terms until the amplitude is less than some percentage (10%, 1%) of the fundamental.

2.2 Exponential Fourier series

By Euler's identity, each sine or cosine term of the trigonometric series of Equation 11, can be written as the sum of two exponential functions. That is, using

$$\cos n\omega_1 t = \frac{e^{jn\omega_1 t} + e^{-jn\omega_1 t}}{2}, \quad \text{and} \quad \sin n\omega_1 t = \frac{e^{jn\omega_1 t} - e^{-jn\omega_1 t}}{2j},$$

each cosine term in Equation 11 converts to two exponential terms as does each sine term. Each pair of exponentials has one term with a $+jn\omega t$ exponent, and one with a $-jn\omega t$. If we collect these exponential terms in order of increasing n exponent, and assign new coefficients, the result is Equation 14.

$$x(t) = \cdots + \mathbf{X}_{-2}\, e^{-j2\omega_1 t} + \mathbf{X}_{-1}\, e^{-j\omega_1 t} + \mathbf{X}_0 + \mathbf{X}_1\, e^{j\omega_1 t} + \mathbf{X}_2\, e^{j2\omega_1 t} + \cdots$$

$$\text{or,} \quad x(t) = \sum_{n=-\infty}^{+\infty} \mathbf{X}_n\, e^{-jn\omega_1 t} \tag{14}$$

where the X coefficients may be complex numbers. (Note that because of their origins with the trig series and the Euler identity, each Xn and X-n pair must be complex conjugates.) The formula for each X n in terms of an and bn is:

$$\mathbf{X}_n = \frac{a_n - jb_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \angle \left[-\tan^{-1} \frac{b_n}{a_n} \right] \tag{15}$$

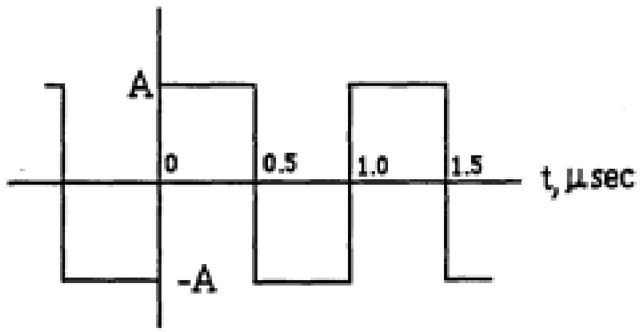
Again, the period and fundamental frequency are related by Equation 12. The constant terms of Equations 11 and 14 represent the average or "DC" value of the periodic function. It should be clear that X0 = a0.

To get a formula for X n, substitute into Equation 15 from Equations 13a or 13b. The result can be expressed in either of the two forms given as Equations 16:

$$\boxed{\begin{aligned} \mathbf{X}_n &= \frac{1}{T} \int_T x(t)\, e^{-jn\frac{2\pi}{T}t}\, dt \\ \mathbf{X}_n &= \frac{1}{2\pi} \int_{2\pi} x(\theta)\, e^{-jn\theta}\, d\theta \end{aligned}} \tag{16}$$

where again, $\theta = \omega_1 t$ and $\omega_1 T = 2\pi$

Again using the square wave of the previous example, calculate the expnential Fourier series approximation:



Again, we change the time axis to the $\omega_1 t = \theta$ axis. To obtain the $\mathbf{X}$n coefficients, use the second form of Equation 16 with the limits of integration $-\pi$ to $+\pi$:

$$\mathbf{X}_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x(\theta)\ e^{-jn\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{0} (-A)\,e^{-jn\theta}\,d\theta + \frac{1}{2\pi}\int_{0}^{+\pi} (A)\,e^{-jn\theta}\,d\theta$$

Each integral yields two terms:

$$\mathbf{X}_n = \frac{A}{2\pi jn}\left[1 - e^{jn\pi} - e^{-jn\pi} + 1\right]$$

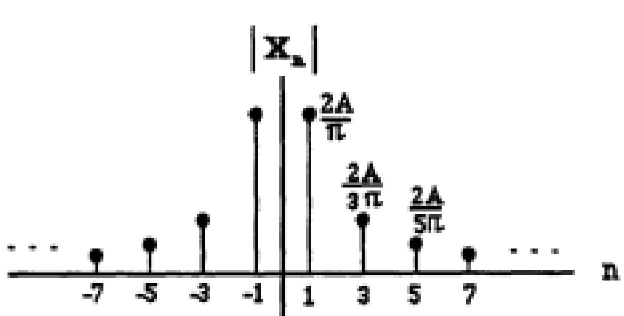
The quantity in the brackets is worth 4 for n odd, and zero for n even. Therefore, the Fourier series representation of x(t) is

$$x(t) = \cdots + \frac{2jA}{5\pi}e^{-j5\omega_1 t} + \frac{2jA}{3\pi}e^{-j3\omega_1 t} + \frac{2jA}{\pi}e^{-j\omega_1 t} - \frac{2jA}{\pi}e^{j\omega_1 t} - \frac{2jA}{3\pi}e^{j3\omega_1 t} + \cdots$$

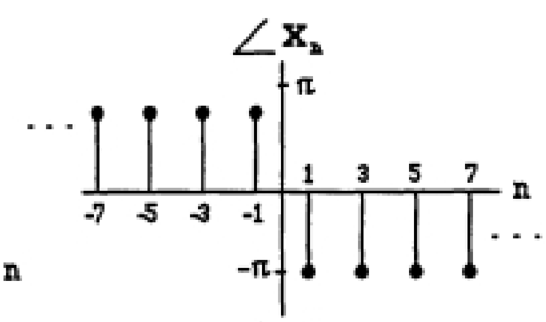
Note that by combining fundamental terms, third harmonic terms, etc., this series can be transformed into the trigonometric series previously determined for the square wave.

2.3 Spectrum Plot

To show the amplitude or phase angle of each term of the Fourier series graphically, the X n values (magnitude and phase) or the magnitude and phase of the trigonometric coefficients are sometimes plotted vs ω or vs. harmonic number, n. For example, the spectrum plot of the square wave just solved is given below, using magnitude and phase of the $\mathbf{X}_n$ coefficients.



Magnitude



Angle

The same kind of plot can be done using the trigonometric coefficients. In this case, only positive values of frequency apply. The magnitude of each term is $\sqrt{(a_n{}^2 + b_n{}^2)}$. The phase angle is $-\tan^{-1}(b_n/a_n)$.

2.4 Symmetry Considerations

The determination of the Fourier coefficients can sometimes be simplified by the observation of certain rules which result from the symmetry of the waveform. There are two important symmetry considerations if the trigonometric series is being developed: odd-even symmetry and half wave symmetry.

Odd-Even symmetry:

Even function: A function is even if x(t) = x(-t). An even function can be folded on itself about the t = 0 axis. A cos wt is an even periodic function'

Odd function: A function is odd if x(t) = - x(-t). An odd function looks exactly the same if its graph is rotated 180°. A sin cot is an odd periodic function.

Figure 3 illustrates some odd and even periodic functions. The functions on the top row are even and those on the botom row are odd. Note that an odd function must have an average value of zero. You should be able to recognize these special properties from the graph of a function. Most periodic functions are neither even or odd. However, any periodic function can be expressed as the sum of two other periodic functions, one odd and one even.

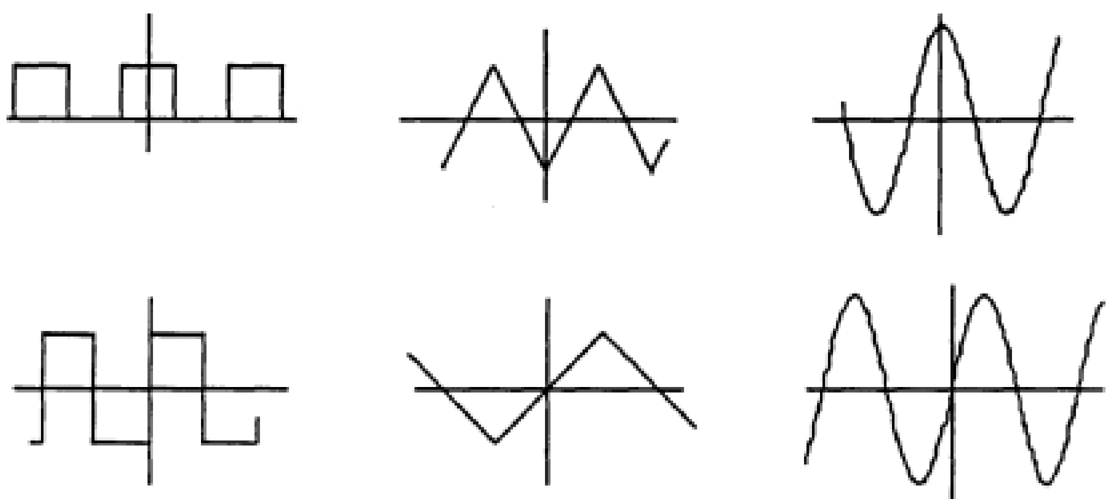


Figure 3 Even (top row) and odd (bottom row) periodic functions.

The reason for our interest in the even or odd qualities of a periodic function is that by recognizing these properties we can simplify the trigonometric Fourier series determination. If an even function is written as a trigonometric Fourier series, all terms of the series must be even (cosine terms only), since the introduction of even one sine term would cause unequal values to be added at points t and -t, destroying the eveness of the sum' Similarly, if an odd function is expressed as a Fourier series, the series must contain only odd terms (sine terms only), because the introduction of a cosine term would add equal values at times t and -t, destroying the odd property of the sum. Therefore, we can state the following rules for the determination of the a and b coefficients of a trigonometric Fourier series:

The Fourier series expansion of an even function contains no sine terms (bn = 0).

The Fourier series expansion of an odd function contains only sine terms (an = 0).

Returning to our previous square wave example, we see that since that wave was odd, we could have expected all the an terns to be zero.

Half-wave symmetry

A periodic wave has half-wave symmetry if x(t) _= - x(t + T/2). Each half cycle of the wave is repeated in the next half cycle with reversed sign. Sine and cosine waves are examples of waves with half-wave symmetry. In Figure 1, only waves B and D have half-wave symmetry. In Figure 2, none of the waves has half-wave symmetry. In Figure 3, all but the first wave have half-wave symmetry. 'The property of eveness or oddness is separate from half-wave symmetry as indicated by Figure 4.

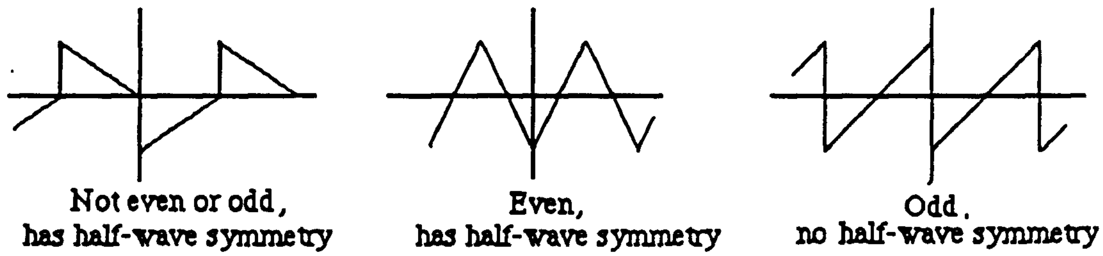


Figure 4 Examples which illustrate that half-wave symmetry is independent of odd or eveness.

Suppose we watch two sinusoids, one at the fundamental frequency, and the other an odd harmonic, starting at any point on the time axis. After one half cycle of the fundamental, both waves have completed an odd number of half cycles. Therefore, both are about to repeat their waveforms of the previous T/2 interval, but with reversed sign. If we add these sinusoids, the waveform of the sum during the first T/2 will be identical to that of the sum during the second T/2, but with the sign reversed. The sum is therefore a wave of period T which has half-wave symmetry. The same is true no matter how many odd harmonics are summed. Now suppose the second wave is an even harmonic. After a half cycle of the fundamental, the even harmonic has completed some number of full cycles and is therefore about to repeat its waveform with no sign change. When equal values are added to the fundamental during each half cycle, half-wave symmetry will be destroyed.

This is illustrated further in Figure 5. Note that the sum of the fundamental and the second harmonic (left panel) does not show half-wave symmetry, but the sum of the fundamental and the third harmonic (right panel) does possess half-wave symmetry.

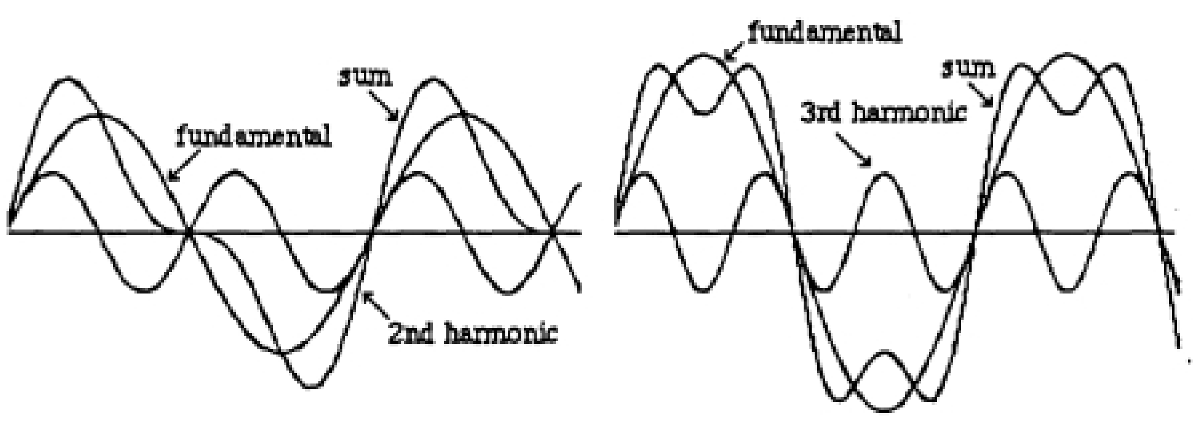


Figure 5. Presence of any even harmonic destroys half-wave symmetry.

From the above discussion, we can formulate the following rule:

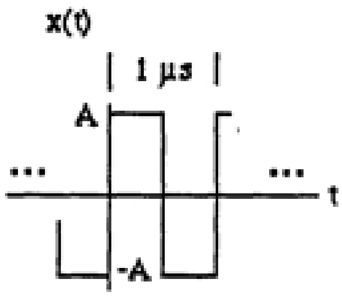
If x(t) has half-wave symmetry, its Fourier series contains no even harmonics.

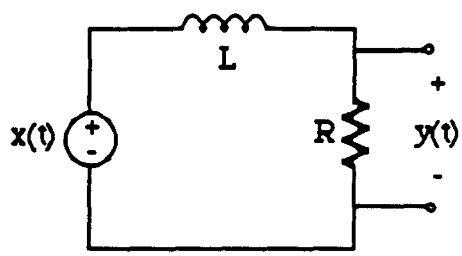
Applications of Fourier Series

We have studied techniques for determining the response of a linear system to constant or sinusoidal forcing functions. However, if x(t) is periodic but not sinusoidal, the response to this input may be difficult to find directly. If x(t) is approximated by a number of Fourier sinusoidal terms, the response to each of these terms may be found separately using sinusoidal analysis. Then, by superposition, these responses can be added to approximate the system response to x(t).

The transfer function of a system, H (s), can be obtained using standard analysis techniques. $\mathbf{H}(j\omega)$ can be used to find the system's response to sinusoidal driving forces. For any frequency, say $n\omega_1$, $\mathbf{H}(jn\omega_1)$ is a complex number which multiplies the input phasor to give the output phasor. That is, the magnitude of each term of the input Fourier series is multiplied by the respective $|\mathbf{H}(jn\omega_1)|$ and its phase is shifted by an amount equal to the angle of $\mathbf{H}(jn\omega_1)$. Following are some examples which illustrate this technique.

Given that x(t), the input voltage to the electric system below, is the square wave of the previous examples, determine the Fourier series approximation of the output, y(t). The elements of the system are R = 10KΩ and L = 2.5 mH.





We choose to do this problem using the trigonometric series. We already have found that x(t) can be expressed as

$$x(t) = \frac{4A}{\pi}\left[\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

To determine y(t), we must use superposition and apply x(t) term by term to the system. The system transfer function is different for each harmonic since it is a function of the frequency:

$$\mathbf{H}(j\omega) = \frac{\mathbf{Y}(j\omega)}{\mathbf{X}(j\omega)} = \frac{R}{R + j\omega L}$$

Substituting the R and L values and replacing ω by $n\omega_1$ where $\omega_1 = 6.283\times10^6$ r/s, we get

$$\mathbf{H}(jn\omega_1) = \frac{1}{1 + j\,1.57\,n}$$

This expression represents the attenuation (magnitude of H) and phase shift (angle of H) which each harmonic of x(t) will receive as it passes through the system. A quick way to compute attenuation and phase shift values for the first 5 odd harmonics is to use Maple: z:= 1/(I+I*n*1.57); for n from 1 by 2 to 9 do convert(z,polar) od;

("n from 1 by 2 to 9" means step n from 1 to 9 in increments of 2.)

The values Maple provides for transfer function magnitude and angle (rounded) are:

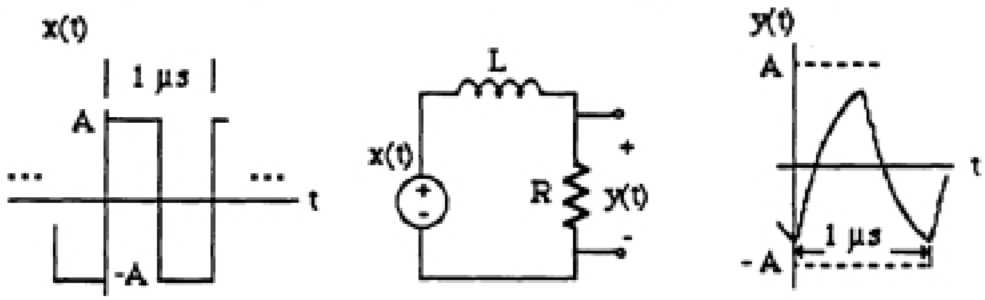
harmonic	magnitude	angle (rad)
1	.537	-1.00
3	.208	-1.36
5	.126	-1.44
7	.0906	-1.48
9	.0706	-1.50

More harmonics could be included, but the attenuation indicates they will not be of significance.

To determine y(t) coming out of the system, we must multiply each term of x(t), and shift its phase according to the table above. The result (first 3 terms) is:

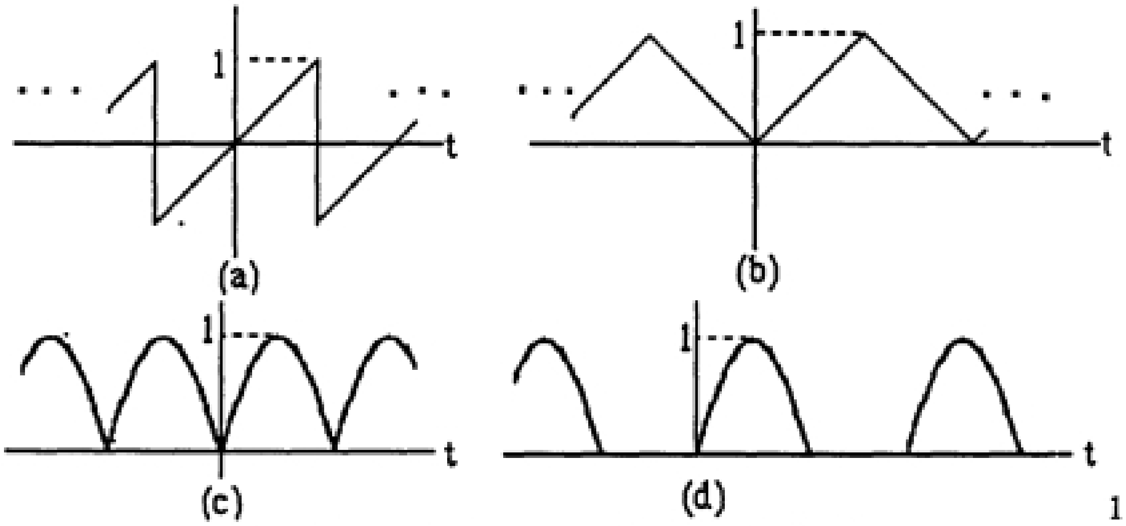
$$x(t) = A\left[0.684 \sin(\omega_1 t - 57.3°) + 0.0882 \sin(3\omega_1 t - 77.9°) + 0.0321 \sin(5\omega_1 t - 82.5°) + \cdots\right]$$

A plot of y(t) including harmonics through the 11th is given on the right, below.



Notice that the effect of the system, which we recognize as a lowpass filter, is to slow the rise and fall of the input signal.

Chapter 16 Problems



In each case, the frequency of the wave is 100 Hz.

1. Find the trigonometric Fourier series coefficient values for waves (a), (b), (c), and (d) above. Write out the first three non-zero terms of each series. Use the methods below: (i) For (a), (b) and (d), set up the required integrals, then integrate using Maple to get the Fourier coefficients. (ii) Repeat (b), this time differentiating first to get a simpler integral. Don't use Maple (iii) Obtain the trigonometric series for (c) by manipulation of the series for (d).

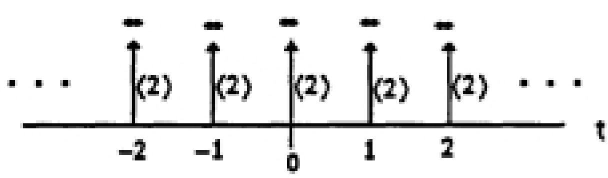
2. For the four waveforms of Problem 1, sketch magnitude and phase angle spectrum plots.

3. Find the exponential Fourier series for (d). Use either the definition or Euler's identity. (ans:

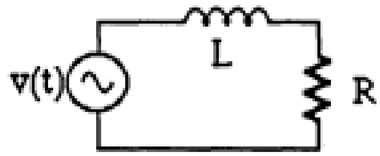
$$\frac{1}{4j}e^{jx} - \frac{1}{4j}e^{-jx} + \sum_{\substack{n=-\infty \\ n \text{ even}}}^{+\infty} \frac{e^{jnx}}{\pi(1-n^2)}$$

where $x = \omega_1 t$))

4. Find the exponential Fourier series for the wave below. Write out 5 terms. (ans all $X_n = 2$).



5. Suppose wave (a) has a fundamental frequency of $\omega_1 = 2000$ r/s. It is the input voltage wave to the RL lowpass filter shown on the next page ($R = 1000\Omega$, $L = 0.5H$). Use Fourier techniques and Maple to obtain a plot of the output voltage waveform (appearing across R). Include a sufficient number of terms for an accurate picture. (Don't forget to adjust both magnitude and phase).



As a guide, the Maple commands to find the a_n and b_n coefficients for wave (a) are given below. Note that these must still be evaluated for each n before the series can be written.

- an := (2/Pi)* int((1/Pi)*cos(n*x),x=-Pi..Pi); (the a_n, of course, should = zero.)
- bn := (2/Pi)* int((1/Pi)*sin(n*x),x=-Pi..Pi);

Some Fourier series results:

Wave (a): $2 \sin \omega_1 t - \sin 2\omega_1 t + (2/3) \sin 3\omega_1 t - ...$

Wave (b):

$$\frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \omega_1 t}{1^2} + \frac{\cos 3\omega_1 t}{3^2} + \frac{\cos 5\omega_1 t}{5^2} + \cdots\right]$$

Wave (c):

$$\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2\omega_1 t}{1 \cdot 3} + \frac{\cos 4\omega_1 t}{3 \cdot 5} + \frac{\cos 6\omega_1 t}{5 \cdot 7} + \cdots\right]$$

Wave (d)

$$\frac{1}{\pi} + \frac{1}{2}\sin \omega_1 t - \frac{2}{\pi}\left[\frac{\cos 2\omega_1 t}{1 \cdot 3} + \frac{\cos 4\omega_1 t}{3 \cdot 5} + \frac{\cos 6\omega_1 t}{5 \cdot 7} + \cdots\right]$$

